APPARENT ADDITIONAL MASS CHARACTERISTICS OF VARIOUS BLUFF BODIES OF REVOLUTION

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APPARENT ADDITIONAL MASS CHARACTERISTICS

OF VARIOUS BLUFF BODIES OF REVOLUTION

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APPARENT ADDITIONAL MASS CHARACTERISTICS

OF VARIOUS BLUFF BODIES OF REVOLUTION

SUMMARY

A simple torsion pendulum was used to make a series of tests to determine the apparent additional mass characteristics of cylinders of revolution with blunt, conical, and hemispherical ends. Data are presented for each configuration for motion parallel, perpendicular, and forty-five degrees to the longitudinal axis. Graphs showing the variation of apparent additional mass and apparent additional mass coefficient with fineness ratio are included along with tabular data. Experimental and theoretical results are compared where possible. The important developments in the study of apparent additional mass are traced briefly from a historical standpoint, basic theory is presented, and suggestions made concerning the need for additional theoretical and experimental data.

The results of the tests conducted in both water and air indicate that the apparent additional mass of a cylinder with blunt ends is greater than that of a similar cylinder with either conical or hemispherical ends. All the curves of apparent additional mass plotted against fineness ratio or length are essentially straight lines. The attitude of a cylinder with respect to its direction of motion has a great effect on its apparent additional mass. It appears that apparent additional mass coefficients become large and tend to approach a constant value as the fineness ratio approaches 10 for the bodies tested. The

coefficients ranged in magnitude from values greater than 0.2 for motion parallel to longitudinal axes to almost 1.8 for motion perpendicular to longitudinal axes.

Theory based on perfect fluid assumptions predicts a value of 1 for the apparent additional mass coefficient of a cylinder moving perpendicular to its longitudinal axis in water. As a result of viscosity effects experimental values greater than 1 should be expected. This was confirmed by the tests. No other theoretical data were available for comparison.

The torsion pendulum appears to be a reliable and convenient apparatus for determining the apparent additional mass characteristics of the bodies tested.

It is suggested that additional tests be made to determine more exact characteristics for low fineness ratios, say from 0 to 4. Very little information on scale and velocity is available. Investigations of the effects of wall boundaries and polished surfaces, and tests to check the meager experimental results available could contribute much to this field of study.

INTRODUCTION

When an object is set in motion in a fluid the surrounding fluid is also set in motion. It is an important fact, yet not too well recognized, that the effective mass of an object in a fluid will be greater than in a vacuum. There is nothing to hinder the motion of a body through a perfect vacuum once it has started, and the same is true of a nonviscous, incompressible fluid. However, should a body be accelerated in air or some other fluid, perfect or viscous, the force required would be that to accelerate the mass of the body plus a mass of fluid affected by the body. This affected mass of fluid is the source of the apparent additional mass present in numerous practical cases of accelerated motion. An increase in both moment of inertia and mass must be accounted for in engineering calculations. Numerical values of these quantities are not always negligible, and in some cases such as hollow bodies in water, apparent additional quantities may greatly exceed the static or dead weight.

The primary purpose of this paper is to extend the experimental data concerning the apparent additional mass characteristics of various bluff bodies or revolution. To reach this objective it is necessary to trace briefly from a historical standpoint the important developments in the study of apparent additional mass, to present the basic theory, to define the terms used, to summarize the results of experiments performed by the author to determine the apparent additional mass characteristics of certain bluff bodies, and thereby to provide a background for more extended experiments in this field.

The determination of apparent additional mass characteristics is important in calculations involving:

1. boats
2. dirigibles
3. maneuvering of airplanes
4. missiles (rockets, bombs, etc.)
5. pendulum tests
6. spinning of airplanes
7. submarines
8. vibration tests
9. wind tunnel test corrections where longitudinal static pressure gradients exist.

Very little has been written on either theoretical or experimental determination of apparent additional mass, and experimental methods are not well developed. The torsion pendulum has been used in previous investigations and was used by the author to obtain much of the experimental data presented in this paper. Bluff bodies of revolution were used in the experiments due to the interest of the author and the practical possibilities of changes in length and end configurations. By selecting an end configuration for a given cylinder and systematically changing the length of the cylinder it is possible to obtain a series of correlated test data for numerous investigations. Fairly adequate data for spheroids are available from previous tests and theoretical treatment is well developed.

It is important to point out that when a body is accelerated in a fluid, each particle is not necessarily accelerated in the direction of motion, nor is the apparent additional mass always a simple function of the displaced fluid. As an illustration, theoretically the apparent additional mass of a sphere is one-half of the mass of the displaced fluid. For a somewhat more complicated body, say a dirigible, the apparent additional mass is taken as about equal to the mass of the displaced air. In most cases either experimental results or empirical formulas were relied on to furnish the limited data available. The magnitude of the apparent additional mass of a particular body is a function of its size, shape, direction of motion, density of the fluid, and perhaps other factors such as the extent of the fluid and the extent of nearby objects. Viscosity causes some apparent additional mass, but it is not possible to separate the viscous effects from the total apparent additional mass. Theoretical computations do not agree with experimental data, due largely to the neglect of viscous effects of the fluid in perfect fluid theories. Since theoretical calculations of the potential function for other than simple shapes is very complicated, an empirical approach to the computation of the apparent additional masses of such shapes is necessary both to confirm theory and furnish data for engineering applications. Therefore it appears that experimental rather than theoretical data must be relied on in practice.

To appreciate thoroughly the importance of both experimental and theoretical progress in this field it is necessary to realize that the dynamical treatment of any mechanical system, submerged in part or entirely in a fluid, must take into account the apparent additional mass of the submerged parts. An investigation of bluff bodies of revolution is particularly pertinent to missile research and to other problems in aerodynamics of special interest at this time.

HISTORICAL SKETCH

It is believed that scientists have been aware of apparent additional mass longer than historical records indicate. However, the first written evidence that a body could possess an apparent additional mass was recorded late in the seventeenth century. A historical summary of early developments in apparent additional mass studies is not readily available in most libraries. Towsley[1] reproduced Green's paper from the Transactions of the Royal Society of Edinburgh,[2] and much of the historical discussion in this thesis is based on the same source. Less than 200 years ago (about 1787) Chevalier Du Buat found that a sphere must have an apparent additional mass to account for its behavior while oscillating in water. Later in 1828 Bessel confirmed Du Buat's discovery by studying the behavior of a seconds pendulum. He found that it was necessary to use a correction factor to account for an apparent increase in inertia. By expressing the increase by *k* times the mass of displaced fluid he was able to establish values of *k* for spherical pendulums in water and in air. It is of interest to note that it was a pendulum that led to the discovery of the existence of apparent additional mass. Du Buat established $k = 0.585$ by experimentating with spheres in water. Bessel found *k* to be 0.6 for water and 0.9 for air.

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[1] Melvyn F. Towsley, "Apparent Additional Mass," (unpublished thesis in partial fulfillment of the requirements for the degree, Master of Science in Aeronautical Engineering, Daniel Guggenheim School of Aeronautics, Georgia School of Technology, Atlanta, June, 1947), p. 4.

[2] George Green, "Researches on the Vibration of Pendulums in Fluid Media," Transactions of the Royal Society of Edinburgh, 13:54-62, 1866.

The magnitude of these values must have indicated that the effects of apparent additional mass could be of practical importance in engineering applications.

While investigating the effects of air and hydrogen on the periods of vibrating pendulums Sabine observed that the times required to damp the vibrations were in the ratio of 1 to 5.25, which is not the ratio of the density of hydrogen to that of air. It was expected that the ratio would be the same as the ratio of the two densities, so the results indicated the presence of apparent additional mass. Later Baily confirmed Sabine's observations, using several sizes of cylindrical and spherical bobs. No one had offered a theoretical solution to the problem, and Poisson was probably the first to suggest a mathematical approach.

Poisson suggested that by neglecting the viscosity of the fluid the value of k for a sphere could be established to be 0.5. Other authors, Green, Plana, Stokes, and Lamb used various methods to establish k in agreement with Poisson. At this point there was agreement concerning the fact that since fluid adheres to a body the density of the fluid could affect the apparent additional mass. It was convenient to neglect the density in making a theoretical approach.

At this point Stokes suggested that the viscosity of a gas could be determined by pendulum oscillations, and his experiments were fairly successful. He found that the value of k used in Poisson's formula should be increased. This correction indicates closer agreement with the values found by Baily and Du Buat. Stokes found that it was rather difficult to check the viscosity of water by the pendulum method, his major problem being the difficulty of making the necessary correct observations of the pendulum in water.

Several others contributed to findings in this field. McEwen obtained better agreement with his pendulum experiments in oil and water. Later experiments by Barnes, McEwen, and Krishnayar made use of oscillating spheres. A value of $k = 0.46$ was obtained by Cook, who dropped a large sphere down a mine shaft. Cook's value of $\underline{k}$ was less than the values obtained by previous investigators. Stokes' theory indicated $k = 0.530$, 0.536, and 0.530. Krishnayar obtained values of 0.584, 0.585, and 0.580. An important conclusion was that Du Buat's value of 0.585 and the other empirical values greater than 0.500 for spheres indicated that the theory based on fluid adhering to the surface of the body was correct.

Up to this time interest in the subject had not been great, due perhaps to the lack of need for the knowledge in engineering applications. What is now considered an important case was that of an ellipsoid in an infinite fluid worked out by Green and Lamb. Taylor confirmed the suggestions that a rigid boundary increases inertia. Stokes had found that $K = 1$ for a long cylinder moving in water perpendicular to its axis. He concluded that boundary effect is not negligible. There appeared to be not great need for intense study until the development of the airship. Stability and performance problems created appreciation of the importance of both theoretical and practical apparent mass studies.

Green's paper was published in 1836, and it could be considered as the most important of the early works on the subject. Much of the later work in apparent additional mass studies is based on Green's fundamentals. Since Du Buat's discovery about 160 years ago not a great deal has been accomplished, but the importance of the problem is now realized, and an increasing interest is evident. Since World War I several papers concerning apparent additional mass have been published; however, a great deal remains to be done. It is realized that apparent additional mass effects must be accounted for in all types of accelerated motion, and that in many cases these effects are considerable in magnitude. Dr. Max M. Munk is one of the most widely known authors in the field of apparent mass studies in this century.

It is of interest to note that the effects of apparent additional mass in a compressible fluid have yet to be investigated. A great deal remains to be done in approaching the subject from perfect fluid theory, and it appears that there probably will be considerable lag between fundamental research and practical needs.

DEFINITION OF TERMS

The term "apparent additional mass" is probably the most descriptive of the various expressions applied to the quantity with which this paper is concerned. Practically all authors who have contributed to this field have used their own definitions. This paper is no exception, but there is no departure from the basic meaning of the definitions used. All definitions and terms applied to apparent additional mass deal with a quantity which does not directly exist but acts as a real mass. Such expressions as "apparent mass," "entrained mass," "virtual mass," and "additional mass" appear in literature on the subject. The term "apparent additional mass" is used consistently in this paper.

It is important to note that the expression "entrained mass" should not be applied to apparent additional mass. The two masses are difficult to separate, and in experimental work it is impossible to account for "entrained mass" effects, but in no sense are they the same mass. Unfortunately, entrained effects are especially prevalent at the low velocities and accelerations at which most experimental work is done.

It is convenient to apply the term "apparent additional" to other quantities based on a mass concept, such as momentum and moment of inertia. Definitions and nomenclature for some of the basic terms in common use are as follows:

I_A = the apparent additional moment of inertia

$I_o = I_S$ = the true moment of inertia as measured in a vacuum

k = the apparent additional mass coefficient and is equal to the apparent additional mass divided by the mass of the fluid displaced ($m_A/m_{D.F.}$)

k' = the apparent additional moment of inertia coefficient and is equal to the moment of inertia of the apparent additional mass divided by the moment of inertia of the displaced fluid ($I_A/I_{D.F.}$)

K = a special apparent additional mass coefficient, used in equations of the form $m_A = K \times C$, where C is a constant dependent on the body dimensions

I_v = the virtual moment of inertia or the moment of inertia as measured in a fluid (not a vacuum)

m_A = the apparent additional mass

ρ = the mass density (unless otherwise defined, in slugs/ft^3)

Other nomenclature is presented as necessary in the development of the formulas and expressions used in the report.

THEORY

There is a fundamental difference between the motion of a body in a perfect fluid and its motion in a vacuum. The fluid surrounding a body is affected by the motion of the body, and this effect is subject to theoretical treatment. In general, the theoretical treatments by recognized authors are similar in fundamentals but vary somewhat in details and analogies.

In the case of a body moving through an infinite, motionless field of fluid particles there is work done on each particle as the body replaces the fluid particles in its path. Work must be done in moving a particle of definite mass, and as the body moves there is an instantaneous flow created similar to the result of impulse. The impulse of a force may be defined as the product of the force and the time interval during which the force acts if the time interval is very small, and each increment of the body surface creates an impulse, the direction of which is normal to the surface increment.[3]

The usual conception of kinetic energy may be used to identify apparent additional mass. Since impulse is equal to the momentum imparted to each particle, and kinetic energy is equal to one-half the product of the square of the velocity and the mass, kinetic energy is equal to one-half the product of the momentum and the velocity. In other words, the change of the kinetic energy of the fluid due to the presence of the body is proportional to the product of an impulse times

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[3] Towsley, loc. cit.

the normal velocity, and the momentum of the fluid may be substituted for impulse. Apparent additional mass may be obtained from the kinetic energy equation by dividing it by $\frac{1}{2}V^2$. A body moving with uniform velocity constantly creates momentum at one point and destroys it at another, causing the total momentum in a perfect fluid to remain constant. During uniform acceleration of a body in a fluid momentum is created and stored in the fluid. Since it appears that the kinetic energy is a function of the velocity normal to the surface and the impulse applied by each surface increment, the calculation of the kinetic energy should make it possible to determine an expression for the apparent additional mass.

Dr. Max M. Munk[4] reasons along similar lines. He states that, although the fluid particles move in a very complicated manner, the motion of each particle takes place as though caused by a mechanical connection between it and the solid, enforcing a motion of the fluid at that point in a certain direction and geared up or down in a certain ratio. An important conclusion can be made concerning the kinetic energy. Keeping in mind that the kinetic energy of each particle is proportional to the square of its velocity, and that its velocity is proportional to the velocity of the solid, the energy of each particle is proportional to the square of the velocity of the solid and also to the kinetic energy of the fluid as a whole. The same conclusions can apply to the momentum of the fluid. The momentum is proportional to

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[4] W. F. Durand, <u>Aerodynamic Theory</u>, Vol. 1 (Berlin: Julius Springer, 1934), pp. 224-304.

the velocity of the solid, but not necessarily parallel to it. The desired momentum is that imparted by the force necessary to create the motion. The sum of all momenta of the individual particles would become mathematically indefinite were its evaluation by integration over the space attempted.

The energy concept indicates that a solid moving with constant velocity under no applied forces cannot experience drag parallel to its motion. If drag were present there would be a continual consumption of energy without an increase in the fluid energy. Of course, this would be contrary to the law of conservation of energy. This applies to the parallel component of force, but other components of the resultant fluid force could exist. The result of this thinking is justification of the perfect fluid concept for apparent additional mass studies.

For a uniformly accelerated motion of a solid in a fluid there is an external force necessary for the acceleration of the mass of the solid plus an additional force for whatever mass of fluid is accelerated along with that of the body. The entire fluid must be considered as taking part in the motion, but particles far away move very slowly and both the accelerating force and the energy developed are finite. The energy of a solid of mass m may be expressed as follows:

$$K.\,E. = \tfrac{1}{2}\,AU^2 ,$$

where $K.\,E.$ = kinetic energy ,

$A = m$ = mass ,

and U = uniform velocity (magnitude of the velocity of the fluid).

The kinetic energy and force necessary to accelerate the solid plus its surrounding fluid (perfect fluid) can be computed by adding to the actual mass of the solid a fictitious mass. This fictitious mass may be called the "apparent additional mass" of the solid or any other appropriate designation. The inertial forces of the fluid are proportional to its density, which is usually assumed constant. Therefore the apparent additional mass of the body is equal to the density of the fluid multiplied by a volume. This volume may be called the "apparent additional volume" or the volume of the apparent additional mass. It should be emphasized that this volume depends on the geometrical outlines only, including its position relative to the direction of motion.

A theoretical approach to the subject would be to establish an equation for the computation of apparent additional mass and the volume of the mass. First, it would probably be best to compute the kinetic energy of the fluid involved and from these computations the apparent additional mass. It is convenient to consider that the motion is created from rest. Bernoulli's pressure equation may be used to advantage at this point.

For irrotational motion and no external forces:

$$\rho\left(\frac{\partial}{\partial t}\nabla\varphi - \tfrac{1}{2}\nabla U^2\right) = \nabla p ,$$

$$\rho\left(\frac{\partial\varphi}{\partial t}\right) - \frac{\rho U^2}{2} = p + \text{constant}.$$

For a steady flow:

$$p = -\frac{\rho U^2}{2} + \text{constant},$$

where

$$\nabla = \text{del} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

= a vector operator or differentiating symbol,

φ = scalar potential of the velocity field (usually just potential or velocity potential),

$\nabla\varphi$ = antigradient (referred to as gradient by some authors),

$-\nabla\varphi$ = gradient (referred to as antigradient by some authors),

t = time,

U = magnitude of the velocity of the fluid,

and p = pressure.

The following fundamentals apply:[5]

Fluid velocities are vectors having not only magnitude but also direction.

Three scalars are necessary to fully describe velocity.

The velocities at all points form the vector field of the fluid velocity.

The properties of a vector field at a particular point may be analyzed by considering the velocity relative to a geometric conception of a small plane element having area and direction (a vector).

V = surface velocity (a vector quantity) = $iu + jv + Kw$.

dS = a vector surface element of area.

Flux = $dS \cdot V$.

$\nabla \cdot V$ = divergence of velocity,

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

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[5]Alan Pope, "Basic Wing and Airfoil Theory," (unpublished advanced aerodynamics notes, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, 1948), pp. 1-23.

$= i \cdot \frac{\partial V}{\partial x} + j \cdot \frac{\partial V}{\partial y} + k \cdot \frac{\partial V}{\partial z}$,

= volume of fluid leaving a small closed space element of unit volume per unit time, passing through its surface from inside to outside.

$\nabla \times V$ = rotation or curl of velocity,

= twice the mean angular velocity of the fluid element,

$= i \times \frac{\partial V}{\partial x} + j \times \frac{\partial V}{\partial y} + k \times \frac{\partial V}{\partial z}$.

x, y, z = Cartesian coordinates defined in the usual sense.

i, j, k = unit vectors related to Cartesian coordinates in the usual manner.

r = vector from origin of Cartesian coordinates to point in space.

$x = i \cdot r$.

$y = j \cdot r$.

$z = k \cdot r$.

$\nabla \cdot \nabla \times V = 0$.

$\nabla \cdot \nabla \varphi = 0$.

∇p = force per unit volume.

At the points on the surface of the solid the pressure term corresponding to the steady motion does not perform any work. Of the expression

$$\rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho V^2 = p + \text{constant},$$

only the term $\rho(\partial \varphi / \partial t)$ remains. The term $\partial \varphi / \partial t$ is constant for constant acceleration, and the potential is proportional to the time. Its average value is therefore half its final value during the time interval in which the motion is being built up. The energy passed into the fluid through its surface element dS is

$$-\frac{\rho}{2}\,\phi\,\nabla\phi\cdot dS\,.$$

This expression may be integrated to obtain the entire kinetic equation

$$\begin{aligned}\text{K. E.} &= T\\ &= -\frac{\rho}{2}\int\phi\,\nabla\phi\cdot dS.\end{aligned}$$

It is necessary to integrate over the entire surface of the solid, but the entire energy is obtained, because at a great distance from the surface there is no motion, and any variation of pressure at a great distance cannot cause the transfer of energy from the distant points.

By a transformation of the kinetic energy of each particle the above integral can be obtained directly. Using Gauss' transformation, the integral $T = -\frac{1}{2}\rho\iint\varphi\frac{\partial\varphi}{\partial n}\,dS$ can be written as a divergence and transformed from a space integral into a surface integral.

Let φ and ψ be any two scalars satisfying

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

$$\nabla\cdot(\varphi\nabla\psi)=\nabla\varphi\cdot\nabla\psi+\varphi\nabla\cdot\nabla\psi=\nabla\varphi\cdot\nabla\psi$$

If $\varphi=\psi$, then

$$\nabla\cdot(\varphi\nabla\varphi)=\nabla\varphi\cdot\nabla\varphi.$$

This gives:

$$\frac{\rho}{2}\int \nabla\varphi\cdot\nabla\varphi \, dQ = \frac{\rho}{2}\int \nabla\cdot(\varphi\nabla\varphi) \, dQ$$

$$= \frac{\rho}{2}\int \varphi\nabla\varphi\cdot dS.$$

The volume of the apparent additional mass may be obtained by dividing the kinetic energy by the dynamic pressure $v^2(\rho/2)$ of the velocity of the solid body so that

$$K = \text{volume of apparent additional mass} = \frac{2\,T}{U^2}\,.$$

A non-dimensional quantity depending only on the shape of the body and its direction of motion may be obtained by dividing the volume of the apparent additional mass by the volume of the solid.

k = coefficient of apparent additional mass or often the "inertia factor"

This coefficient becomes infinite as the volume of the solid approaches zero without the kinetic energy becoming zero. Of course, in this case the definition is not valid, as in the case of an infinitely thin disk moving normal to its plane.

It is important to note that there is a fundamental difference between the motion of a solid in a vacuum and its motion surrounded by a perfect fluid. The apparent additional mass of a solid surrounded by a perfect fluid may, in general, vary with direction of motion. In a vacuum the mass of a solid is equal regardless of direction of motion.

Dr. Munk[6] suggests a method for the theoretical determination of

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[6]Durand, op. cit., pp. 255-258.

apparent additional mass of several simple bodies. Using the sphere as an example, and making use of the superposition of a doublet and a horizontal flow of constant velocity he determined that the apparent additional mass of a sphere should be 0.5 of the mass of the displaced fluid.

For a doublet

$$\varphi = \frac{M}{4\pi r^2} \cos \theta ;$$

for a horizontal flow of constant velocity,

$$\varphi = U\,x = U\,r \cos \theta ;$$

for the combined flows,

$$\varphi = U\,r \cos \theta + \frac{M}{4\pi r^2} \cos \theta ;$$

where

M = constant product of the strength of the source and sink and the distance between them;

x = distance between the source and sink;

r = radius of the sphere formed by the doublet and steady flow.

The spherical shape resulting from the superposition may be verified by computing the components of the velocity represented by the potential normal to the surface of the sphere. Hence

$$-\frac{\partial \varphi}{\partial r} = -U \cos \theta + \frac{M}{2\pi r^3} \cos \theta .$$

For $M = + 2\pi r^3 U$ and $r = R$,

$$- \frac{\partial \varphi}{\partial r} = 0 \; .$$

Then

$$R = \sqrt[3]{\frac{M}{2\pi U}} \; ,$$

and therefore

$$M = 2\pi U R^3.$$

For the potential,

$$\varphi = U \cos\theta \left[r + \frac{R^3}{2 r^2} \right] .$$

At a great distance,

$$\varphi = U \cos\theta \frac{R^3}{2 r^2} \; .$$

The kinetic energy of the flow may be computed thus:

$$T = - \tfrac{1}{2} \rho \iint \varphi \frac{\partial \varphi}{\partial n} dS, \text{ where } dS \text{ is a scalar}$$

$$= \frac{\rho}{2} \int \frac{U}{2} R \cos\theta \, U \cos\theta \, 2\pi R^2 \sin\theta \, d\theta$$

$$= \frac{2}{3} \pi R^3 \frac{\rho U^2}{2} \; .$$

This indicates that the volume of the apparent additional mass of a sphere is equal to half its volume, or that the inertia factor is 0.5. This theoretical value is generally accepted, and empirical values of approximately 0.583, 0.584, etc., confirm the theory based on a perfect fluid. It is important to note that the difference between theoretical and empirical values, due to the assumption of a perfect fluid, are considerable.

The fundamental treatment of the subject by Lamb is often referred to by authors and used as a basis for their approach to the problem. Munk's theoretical results, though arrived at in a somewhat different manner, are the same as Lamb's. Actually, the basic theory is the same in both cases, the main differences being in the mathematical treatment of the problem.

Lamb[7] showed that any actual state of irrotational motion of a fluid could be produced instantaneously from rest by the application of a system of impulse pressures. This approach is possible if a single valued velocity potential exists for the motion. As previously discussed, the kinetic energy is dependent on the velocity normal to the surface and the impulse applied by each surface increment. Let

ϕ = the velocity potential for a two dimensional flow

$$= \int_c (u\,dx + v\,dy),$$

where

u = velocity in x-direction,

v = velocity in y-direction,

n = normal direction.

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[7]Horace Lamb, Hydrodynamics, 5th ed. (Cambridge: University Press, 1930), p. 16.

The normal velocity can be found from the velocity potential. For v = 0:

$$\phi = \int_c^x u\, du = X u$$

$$= \text{velocity x distance}$$

$$= \frac{ft}{sec} \times ft = \frac{ft^2}{sec}$$

By taking the partial derivative of ϕ with respect to n, it is possible to obtain the normal velocity, so that

$$\frac{\partial \phi}{\partial n} = \text{normal velocity.}$$

Following Lamb's development, and taking advantage of the existence of a single valued velocity potential, the impulse pressure necessary to start the motion is $\rho\phi$.

$$\rho\phi = \frac{lb}{ft^3} \times \frac{sec^2}{ft} \quad \frac{ft^2}{sec}$$

$$= \frac{lb\text{-}sec}{ft^2} = \frac{\text{impulse}}{\text{unit area}}$$

$$= \text{impulse pressure}$$

$$\text{Impulse} = \text{impulse pressure x area}$$

$$\Delta S = \text{increment of surface area}$$

$$\Delta T = \text{kinetic energy}$$

The work done by an impulse is equal to the impulse times one-half the sum of the initial and final velocities in the direction of the impulse. When the initial velocity is zero, the average velocity is $\frac{1}{2}\frac{\partial \phi}{\partial n}$.

$$\Delta T = \rho\phi\left[\frac{1}{2}\frac{\partial\phi}{\partial n}\Delta S\right] .$$

$$T = -\frac{\rho}{2}\iint_S \phi\,\frac{\partial\phi}{\partial n}\,dS.$$

The negative sign is used to indicate the inner normal, which is the usual convention. As in other cases,

$$m_A = \frac{T}{\frac{1}{2}V^2} = \frac{2T}{V^2} ,$$

$$K = \frac{m_A}{\rho} ,$$

V = free stream velocity of the body.

The same basic equation can be obtained from Green's theorem.[8]

$$\iiint_V\left(\frac{\partial P}{\partial x}\cdot\frac{\partial Q}{\partial x}+\frac{\partial P}{\partial y}\cdot\frac{\partial Q}{\partial y}+\frac{\partial P}{\partial z}\cdot\frac{\partial Q}{\partial z}\right)dV \quad =$$

$$\iiint_V Q\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right)dV = -\iint_S Q\,\frac{\partial P}{\partial n}\,dS .$$

$\frac{\partial P}{\partial n}$ = the directional derivative of P along inner normal.

$P = Q = \phi$ for a single valued velocity potential.

It is assumed that the velocity potential is finite and can be differentiated over the entire region discussed. For an incompressible

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[8] F. S. Woods, Advanced Calculus (Boston: Ginn and Company, 1932), p. 195.

fluid the Laplacian of ϕ must be zero. Thus,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Taking the volume integral over the entire region and the surface integral over the boundary of the region:

$$\iiint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right] dV = - \iint_S \phi \frac{\partial \phi}{\partial n} dS,$$

and multiplying by $\frac{\rho}{2}$:

$$\frac{\rho}{2} \iiint_V \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right] dV = - \tfrac{1}{2} \iint \rho \phi \frac{\partial \phi}{\partial n} dS.$$

$\rho\phi$ = impulse pressure required to cause motion.

$-\frac{\partial \phi}{\partial n}$ = inward normal fluid velocity.

Thus the work done by the impulsive pressure applied by the surface S causes the motion of the fluid, and this work is expressed by the right hand side of the above equation. The left side of the equation contains the velocity components in the x, y, and z directions.

$$\frac{\rho}{2} \iiint \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right] dV =$$

$$\frac{\rho}{2} \left(U^2 + V^2 + W^2 \right) \iiint_V dV =$$

$$\frac{\rho}{2} (\text{Vol.}) \left(U^2 + V^2 + W^2\right) = \tfrac{1}{2} mV^2 = T,$$

or

$$T = -\frac{\rho}{2}\iint \phi \frac{\partial \phi}{\partial n}\, dS.$$

Since a series of blunt cylinders of revolution were tested by the author, it is necessary to discuss a theoretical approach to the apparent masses of these bodies. Two dimensional flow, body in motion, and fluid at rest conditions provide a basis for the development of the required formulas. The basic equation still applies.

$$2T = \rho \iint \phi \frac{\partial \phi}{\partial n}\, dS.$$

For a circular cylinder of infinite length and axis perpendicular to the direction of motion, the velocity potential is

$$\phi = \frac{V\ a^2}{r} \cos \theta ,$$

where

V = velocity of cylinder,

a = radius of the cylinder,

r = distance from the center of the cylinder to any selected point in the field.

Lamb[9] shows that only the surface conditions are of interest when determining the change in kinetic energy due to the presence of a moving body, and the normal velocity of the cylinder is the radial velocity. Where ℓ = length of cylinder and at $r = a$,

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[9] Horace Lamb, Hydrodynamics, 6th ed. (Cambridge: University Press, 1932), p. 76.

$$\phi = V\, r \cos\theta = V\, a \cos\theta$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial r} = \frac{\partial (V\, r \cos\theta)}{\partial r} = V \cos\theta$$

$$\iint_S dS = \int_0^{2\pi} \ell\, a\, d\theta$$

$$2T = \iint \rho\, \phi \frac{\partial \phi}{\partial n}\, dS$$

$$= \int_0^{2\pi} \rho V\, a \cos\theta \cdot V \cos\theta\, \ell\, a\, d\theta$$

$$= \rho V^2\, a^2 \ell \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= \rho V^2\, a^2 \ell \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta\right]_0^{2\pi}$$

$$= \pi a^2 \ell \rho V^2$$

$$m_A = \frac{2T}{V^2} = \frac{\pi a^2 \ell \rho V^2}{V^2} = \pi a^2 \ell \cdot \rho$$

Therefore the apparent additional mass of a circular cylinder perpendicular to its axis is $\pi a^2 \ell \rho$. It is of interest to note that the volume of the apparent additional mass in this case is equal to the volume of the cylinder

$$k = \frac{\pi a^2 \ell \rho}{\pi a^2 \ell \rho} = 1.0$$

$$K = \frac{\pi a^2 \ell \rho}{\rho} = \pi a^2 \ell$$

Numerous other cases are discussed in the literature on the subject --

disks, straight lines, ellipses, ovals, etc. Towsley[10] presents an excellent summary of the works of recognized authors in the field, and points out that in two dimensional flow the streamlines are the same for all confocal elliptic forms of the cylinder. The case of the circular cylinder is therefore a special case of the motion of an elliptic cylinder moving through motionless fluid. This is true even when the section reduces to a straight line joining the foci of the ellipse.

With this in mind it appears that the apparent additional mass of any two-dimensional elliptic body is proportional to the maximum thickness or length perpendicular to the direction of motion. This holds for a straight line and the other conformal elliptic forms of the cylinder. Where t is the maximum thickness perpendicular to the direction, the apparent additional mass per unit length along the axis may be expressed by

$$m_A = \frac{\pi a^2 \ell \rho}{\ell} = \frac{\pi}{4} \rho t^2,$$

since

$$a = \frac{t}{2} .$$

As previously shown, for a circular cylinder of infinite length and moving perpendicular to its axis, the apparent additional mass coefficient is theoretically one. For motion parallel to its axis, the surrounding fluid would be set in motion only by the viscous forces between the body and the fluid. For a perfect fluid there could be no apparent additional mass under these conditions. From a practical

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[10] Towsley, op. cit., pp. 64-70.

standpoint there is apparent additional mass regardless of the direction of motion, and since viscosity plays such an important part in the magnitude of the apparent additional mass, it appears that experiments to determine this effect are greatly needed. It is the purpose of this paper to present the data obtained from testing a series of blunt cylinders of revolution and to interpret the results in the light of theory and existing data. Only limited data from previous tests are available.

SUMMARY OF PREVIOUS EXPERIMENTAL WORK

In general, the motion of a body may be studied from an additional mass standpoint in three ways: by being oscillated as a pendulum, dropped vertically, or towed in rectilinear motion. Various discussions of all three methods have been presented in several papers published in the United States, England, Germany, and Russia. Many of the original papers are not now available, but much of importance has been referred to in reports that are available.[11] It is not possible to state that any one method is best. A better statement is that there are few advantages and many disadvantages associated with all three methods. Since the density of a fluid directly affects the magnitude of apparent additional mass associated with the motion of a body in the fluid, results of tests in various fluids may be extrapolated to other fluids of varying densities. In other words, apparent additional mass is directly proportional to the density of the fluid. This fact greatly reduces the task of collecting experimental data, and any convenient fluid may be used. Since air can be considered a vacuum without introducing appreciable error, and since the differences between the densities of liquids are relatively small, the use of air and some convenient liquid such as water appears justified for experimental purposes.

When models are dropped or towed Reynolds numbers and accelerations are usually very low. Rather elaborate laboratory equipment is

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[11] William Gracey, "The Additional Mass Effect of Plates as Determined by Experiments," U. S. National Advisory Committee for Aeronautics Technical Report, No. 707, 1941, p. 3.

necessary, and tests are usually expensive. A pendulum test can be simple and less costly, but there is the disadvantage of motion in a disturbed fluid. This may be largely overcome by moving the pendulum system rectilinearly. However, as the apparatus becomes more elaborate, both expense and time become increasingly important. The viscous effects in all cases are somewhat uncertain, and it is impossible to account for them. Also, the size of the tank or channel in which experiments are performed is seldom large enough to completely eliminate boundary effects. In spite of all these limitations considerable experimental work has been completed, some of the most important of which is discussed below.

Some of the early experiments (1918-1919) were made to determine the effect of acceleration on the resistance of a streamlined body. Cowley and Levy[12] reported the results of a series of these tests. A small streamlined body of a fineness ratio of 4:1 was dropped into a vertical tank of water. Accelerations of from 0 to 0.2 g. were obtained. The body used in the experiment was hollow and could be filled with mercury to produce the desired acceleration. The equipment was somewhat complicated. An electromagnet was used to suspend and release the body. A lens was used to project light from an arc to a mirror and into the tank. The results of the tests indicated that for a good streamlined body the acceleration effect was no greater than the order of error in the experiment.

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[12] W. L. Cowley and H. Levy, "On the Effect of Acceleration on Resistance of a Body," U. S. Advisory Committee for Aeronautics Reports and Memoranda, No. 612, Vol. 1, 1918-1919, pp. 95-101.

Frazer and Simmons[13] made similar and more thorough experiments in the 65 foot experimental tank of the William Froude Tank Department. A streamlined body, of fineness ratio 4:1 and length of 18.82 inches, and a sphere 6 inches in diameter were towed in the tank. A working section of the towing tank 28 ft. long was available after allowing for a preliminary run and braking. Since for equal Reynolds numbers and standard conditions the velocity in air is 12.8 times the velocity in water, the low water velocities of from 0.6 to 5 feet per second were equivalent to air velocities from 7.7 to 64 feet per second. For the 6 inch sphere used in these tests velocities equivalent to Reynolds numbers of from 2,000 to 7,000 were obtained. It was considered that the tank was large enough to eliminate the effects of the tank boundary. The data indicated that there was little change of resistance with Reynolds number. This showed that apparent additional mass increases only slightly with velocity. Values of apparent additional mass ranged from about 5 to 49 percent of the mass of fluid displaced, or a mean value of about 19 percent. More pronounced apparent mass effects were found for the sphere. This could be predicted. There is less pressure recovery for an aerodynamically inefficient shape such as a sphere than is the case of a streamlined shape. Theory does not predict or take into account the effects of velocity and acceleration on apparent additional mass, and a great deal of experimental work needs to be done before these effects can be predicted.

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[13] R. A. Frazier and L. F. G. Simmons, "The Dependence of the Resistance of Bodies upon Acceleration, as Determined by Chronograph Analysis," U. S. National Advisory Committee for Aeronautics Reports and Memoranda, No. 590, Vol. 1, 1918-1919, pp. 102-121.

The pendulum method of investigating apparent additional mass is comparatively simple. Either forced or natural oscillations may be used. Also, there is much reliable data on the characteristics of a pendulum. Both theoretical and experimental studies of pendulums are among the earliest of the fundamental scientific studies. The period of a pendulum can be calculated when its moment of inertia and constant are known. Adequate formulas are available for calculating the characteristics of all types of pendulums. Both simple and compound pendulums have been used in apparent mass experiments. The simple torsion pendulum shown in Figure 11, Appendix II, was used by the author to collect data for this thesis. It was a simple matter to determine the torsional constant of the wire by counting and timing the oscillations of a weighted arm attached to the wire. By using the formulas developed under the heading "Pertinent Formulas" and collecting the required data it was not difficult to calculate the apparent additional mass of the bodies tested. Damping appears to be almost negligible. Its effect is to increase the period over the theoretical value. The National Advisory Committee for Aeronautics tests indicate that for tests to determine the moment of inertia of airplanes damping has little affect. Yee-Tak Yu[14] tested circular disks mounted on a simple torsion pendulum in water and found that the damping effect was small enough to be neglected. Errors of less than 0.1 percent resulted from damping, and it is doubtful that experimental accuracy was that small.

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[14]Yee-Tak-Yu, "Virtual Masses and Moments of Inertia of Disks and Cylinders in Various Liquids," Journal of Applied Physics, 13:67, 1942.

In experiments to determine the apparent additional mass effects on the moment of inertia of airplanes it is important to account for all the factors which may affect the true mass. Besides the actual mass of the body there are buoyancy of the structure, entrapped air, and the apparent additional mass. This is true for all types of pendulum tests. However, it is only when bodies are relatively light in weight that buoyancy and entrapped air seriously affect experimental results.

It is necessary to assume that in all oscillation tests the displacements are small enough to make valid the assumption that the sines of the angles are equal to the angles. Also, when frequencies are high and amplitudes small the flow about an oscillating body approaches potential flow during translational motion, which must be assumed even for pendulum motion.

Relf and Jones[15] made some tests to determine acceleration effects on airship models. Small one inch diameter models were oscillated in air and water. Both longitudinal and lateral effects were determined, using various fineness ratios. The models were fitted with hemispherical noses and tested with uniform variation of cylindrical length. Pointed noses were also investigated. Results of the tests are in line with theory. Hemispherical ends resulted in greater values of apparent additional mass coefficients than did the pointed ends. The hemispherical ends must have caused the greater disturbance of the surrounding fluid. This group of experiments is of special interest since the

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[15] E. F. Relf and R. Jones, "Measurements of the Effect of Accelerations on the Longitudinal and Lateral Motion of an Airship Model," Advisory Committee for Aeronautics Reports and Memoranda, No. 613, Vol. 1, 1918-1919, pp. 121-127.

author tested cylinders of revolutions equipped with hemispherical and pointed ends. It is of interest to note that the tests by Relf and Jones show that the minimum longitudinal apparent additional mass coefficient occurred at a fineness ratio of approximately 6:1. Coefficients for the lateral position increased with increasing fineness ratio. For a sphere the apparent additional mass coefficient is established as 0.584. Relf and Jones obtained widely varying values for a fineness ratio of one. Since a cylinder equipped with hemispherical ends becomes a sphere at a fineness ratio of one, it is difficult to explain the results obtained by Relf and Jones. However, an average value of 0.75 for the apparent additional mass coefficient for a sphere results, and this average is not far from the established value of 0.584. The theoretical value for a sphere is 0.5, and for an infinite cylinder 1.0.

Several tests to determine the accuracy of previously reported results were made by Gracey.[16] His main objective was to determine the apparent additional mass effects of flat plates, but he included data on the effect of taper ratio. Some of his references to previous investigations are of interest.

A vibrating spring system was used by the Germans in 1930 to test small plates. Four plates of various fineness ratios were investigated and the results plotted. Formulas were included for the extrapolation of the results to higher aspect ratios.

Investigations to determine the apparent additional mass of a

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[16] Gracey, op. cit., 10 pp.

Bristol pursuit airplane were undertaken by the British.[17] These tests were extended to determine effects of model accuracy and pendulum lengths on the apparent additional mass of a flat plate. A balsa plate having a fineness ratio of 7 was suspended on the arm of a compound pendulum. Results were plotted and presented to show the desired relationships. Balsa models, scale 1:20, were made of the Bristol fighter. One model was carefully constructed to scale and another roughly cut out. It was found that the apparent additional moment of inertia of the crude model was as much as 150 percent greater than that of the carefully constructed one. This indicates the necessity for accuracy in constructing models for apparent additional mass tests.

Russian[18] scientists tried to confirm theoretical formulas for the determination of the apparent additional mass of elliptic plates. Small cardboard frames were covered with paper and suspended in air by a bifilar system. Moments of inertia about both axes were found, and correction factors for the theoretical formulas determined.

Somewhat similar tests were made by the N.A.C.A. in 1933. Plates were made of light wooden frames covered with paper. Span-chord ratios of from 2 to 8 were tested. The plates were 1/4 inch thick and 4 feet long.

In 1937 the Germans[19] investigated two rectangular plates made of steel and aluminum tubing. Tests were made of span-chord ratios

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[17] Ibid., p. 3.

[18] Ibid., p. 4.

[19] Loc. cit.

from 0.25 to 8 by using partially covered frames 0.75 by 3.0 meters. All tests were made with and without covering in air.

A series of tests were made in 1940 by the N.A.C.A. A special 54 inch vacuum tank was used so that pressures could be varied from 27 to 4 inches of mercury. A suspension system was used to obtain the desired data. Apparent additional mass calculations were presented for two rectangular plates of aspect ratios 4 and 6. Apparent additional moment of inertia investigations were made on four rectangular plates of aspect ratios 2, 4, 6, and 8 and on two tapered plates of taper ratios of 2.5:1 and 5:1 and aspect ratio 4. The plates were made of aluminum tubing covered with aluminum foil. Spans were 2.0 inches. Yee-Tak Yu[20] used a simple torsion pendulum to collect data for apparent additional moment of inertia and mass calculations for circular disks, cylinders, rectangular plates,[21] and rectangular parallelepipeds. By using carbon tetrachloride, gasoline, and water a wide range of densities could be investigated. Damping was shown to be negligible, and air was considered an adequate substitute for a vacuum. Graphs were presented and comparisons with theory made where possible. Thin-walled brass tubing was used in the cylinder experiments. At the center of each cylinder a thin metal disk was soldered, and the cylinder was equivalent to two cylinders placed base to base. Apparently, an attempt was made to confirm existing theory for the apparent additional mass

[20]Yee-Tak Yu, loc. cit.

[21]Yee-Tak Yu, "Virtual Masses of Rectangular Plates and Parallelepipeds in Water," Journal of Applied Physics, 16:724-729, 1945.

of cylinders, but the usefulness of using hollow cylinders was not explained. However, several empirical formulas were developed from the data collected. Thin disks made of lead were tested in water and in air. It appears that good qualitative results were obtained. All the models tested were relatively small.

It is not the purpose of this paper to discuss in detail all the available literature on apparent additional mass, but it is possible to make a few statements concerning what has been accomplished. The numerous experiments briefly described in this section have pointed the way toward better experimental technique and have furnished some important data. Theoretical formulas and supporting empirical data are available for the apparent additional mass of such shapes as spheres, flat plates and disks, oblate and prolate ellipsoids, a few streamlined bodies, and rectangular parallelepipeds. Airships and airplanes have been tested fairly successfully for apparent additional mass effects. The theory based on perfect fluid does not account for the viscosity effects of the fluid, and tests reveal much variation between theory and practice. The effects of changes of end configurations and fineness ratio need to be carefully investigated. Sufficient data to predict apparent additional mass effects on cylinders of revolution with various end configurations and moving at various attitudes are not available. Most of the data presented by the few authors contributing to this field of study are more qualitative than quantitative. It must be emphasized though, that the concept of apparent additional mass has practical applications whenever there is accelerated motion of a body through a real fluid. In many cases even a crude approximation is better than neglecting apparent additional mass effects. True moments of inertia calculated from swinging tests and vibration investigations must include allowances for apparent additional mass. Within recent years airships and airplanes have been subject to considerable testing to determine apparent additional mass effects, and with some success.

Malvestuto[22] shows that an equivalent ellipsoid could be used to estimate the apparent additional effects of an airplane fuselage. It is usually considered that the body is made up of flat plates and ellipsoids.

For wings and tails it can be considered reasonably accurate to use coefficients applying to flat plates.

For the majority of cases tested it was found that the additional moment of inertia of airplanes was not greater than 25 percent of the true values. Results were based on data obtained from testing forty airplane models. Body axes were used as reference axes, and the tests were made in an N.A.C.A. vacuum tank.

Swinging tests for full scale airplanes can be made or models can be used. The torsion pendulum appears to be one of the most convenient means of testing models of almost any shape. In spite of its limitations, one would hesitate to state that it is inferior to any other known method.

The author has found very little apparent additional mass data on cylinders of revolution. It appears logical that a series of tests

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[22] F. S. Malvestuto, Jr., and L. J. Gale, "Formulas for Additional Mass Corrections to the Moments of Inertia of Airplanes," U. S. National Advisory Committee for Aeronautics Technical Note, No. 1187, February, 1947, 28 pp.

to determine the apparent additional mass of these bodies should include a study of cylinders with blunt ends; and any other end configurations could be included. Of course, all bodies could be tested at any desired altitudes. Considerable testing will be necessary to establish accurate apparent additional mass coefficients and provide background for more advanced study.

DESCRIPTION OF APPARATUS

The torsion pendulum used in testing the bodies described in this paper consisted of a piano wire 0.0312 inch in diameter stretched across a steel frame as shown in Figure 11 of Appendix III. A somewhat larger or smaller wire could have been used. Steel angle iron 1 inch by 1 inch and 1/4 inch thick, was welded into the 20 by 24 inch frame. Two flat pieces of steel 1/8 inch thick and 20 inches long were shaped into base supports and welded to the steel frame. This piece of apparatus was designed to fit into a steel tank 24 inches in diameter and 18 inches deep shown in Figure 13 of Appendix III. The tank was equipped with a drain faucet and was light enough in weight to be handled by one person without difficulty. The steel arm shown at the top of the frame was supported by a piece of steel welded to the frame. A brass thumb screw threaded into the end of the adjustment arm provided a means of adjusting the tension of the wire. Two steel clamps, one at the top and one at the bottom, made it possible to clamp the wire and hold it at the desired tension.

A somewhat heavy brass clamp was first used to support the arms of the pendulum. This brass clamp was later discarded in favor of a much lighter and more effective steel clamp. The brass clamp appears in Figure 14 of Appendix III and the smaller steel clamp in Figure 11 of Appendix III. A long water channel four feet wide was available, making it possible to use either the small 24 inch tank or the larger channel.

The models were constructed from 24 ST aluminum alloy rod. All models were 0.75 inches in diameter and varied in length from 0.75 inches to 7.5 inches, or from 1 to 10 fineness ratios. Three sets of models were constructed, with blunt, hemispherical and pointed ends. See Figures 15, 16, and 17 of Appendix III. The pointed ends made angles of 45 degrees with the horizontal axes. Also, a pair of steel cylinders 1 inch in diameter and 3 inches long were constructed for calibration tests. All models were drilled and tapped to fit 3/16 inch rods threaded on the ends.

Several sets of brass rods 3/16 inch in diameter and from 5 to 8 inches long were threaded on both ends for use as pendulum arms.

A small aluminum mirror about 1/8 inch square was made and glued to the piano wire near the top of the frame. By reflecting on a screen the light from a movie projector, the mirror made it possible to count the small oscillations of the pendulum in water. This was not necessary for most of the tests, since the oscillations could be counted without the use of the mirror.

The only other items of equipment needed were small hand tools -- screwdriver, small level, pliers, large triangle, etc., and a standard stop watch.

PROCEDURE

The apparatus was set up as shown in Figure 11 of Appendix III, and care was taken to tighten and clamp the wire at the desired tension. It was not necessary to measure the tension, but the wire should be tight enough to assure motion only in a horizontal plane.

Next the pendulum arm clamp was tightly screwed against the vertical wire. This operation caused considerable trouble at first, but it was soon realized that a rigid connection was required, and a suitable steel clamp was designed for this purpose. No motion of the wire within the clamp could be allowed, so by using steel instead of brass a stronger and more effective clamp was made possible.

After the pendulum arm clamp had been properly fitted a set of brass arms were selected to produce a distance of nine inches between the wire and the center of gravity of the two steel cylinders constructed for calibration purposes. The nine-inch length was selected as a matter of convenience. Any other convenient length could have been used. The two steel cylinders were fitted as shown in Figure 11 of Appendix III.

The pendulum was made to oscillate through a small arc and the oscillations timed and recorded. As many as 100 oscillations could be counted and timed in air. This was repeated several times so that discrepancies could be checked and an average value of the time for one complete oscillation determined.

Before the torsional constant of the pendulum could be calculated it was necessary to weigh the steel cylinders and the parts making up the pendulum arm-clamp and rods. These data were necessary for calculating the moment of inertia of the pendulum system for use in the formula for the torsional constant.

$$K_T = 4\pi^2 \frac{I_k}{T_k^2} ,$$

where

K_T = torsional constant of the wire,

I_k = moment of inertia of pendulum,

and T_k = period of oscillation of rod of known moment of inertia.

Standard formulas were used in calculating the moments of inertia of the cylinders, rods, and clamp. As shown in Pertinent Formulas, equation (5), and Appendix I - Sample Calculations, it was not necessary actually to calculate K_T, but there is sufficient data for K_T calculations if desired.

After recording the calibration data the steel cylinders were removed from the pendulum arms and replaced with a pair of models. Oscillations in both air and water were timed and recorded. Many readings were taken for each case, usually a minimum of ten. Timing the oscillations in air was not difficult, but in water the amplitudes were so small that it was necessary to take many readings in order to obtain a reliable average. Ten or more oscillations could be counted in water, but some skill was required to count correctly and time them.

Next the two models were removed and the pendulum oscillated in air and in water without the models. Readings of the time and number of oscillations were recorded as for the other cases. These data were necessary for use in calculating tare effects. Since the rod lengths were not changed during any one series of experiments, it was not necessary to repeat tare reading except as a check on previous data.

All models in the three groups were tested as explained above. It was found convenient to test each pair of models at three attitudes without removing them from the pendulum. The distance from the wire to the center of each model was carefully adjusted and measured. A small hand level and a large forty-five degree triangle were used to secure the proper attitude - either parallel, vertical, or forty-five degrees to the horizontal plane. The pendulum with bodies attached was displaced enough so that from ten to a hundred complete oscillations could be counted and timed. This was repeated ten or more times in both air and water to obtain an average value of the period for each test. The apparent additional mass of each cylinder was calculated and curves drawn for each group of models and for motion vertical, horizontal, and at forty-five degrees to the longitudinal axes of the cylinders. See Tables I through IX of Appendix II and Figures 1 through 6 of Appendix III. Also, apparent additional mass coefficients were calculated and plotted as shown in Tables X through XII and Figures 7 through 9.

Before the apparent additional mass coefficients could be calculated it was necessary to obtain the mass of the fluid displaced by each body. This was accomplished by weighing each body and dividing the weights by the specific gravity of 24 ST aluminum alloy. These values were then plotted and faired to check the accuracy of the calculations. The calculated values were correct in most cases. It was particularly important for the masses of the small models to be exact.

The centimeter gram-second system was used for convenience. A very small change in the masses of the small bodies could make considerable differences in the calculated values of the apparent additional mass coefficients.

It was found that the differences of squared values of the periods in air and in water could be used to detect points out of line with neighboring points. Running plots were made as the data were collected, and much time was saved by eliminating sources of error as they occurred. The formulas used and their development are presented under the section "Pertinent Formulas." In general, the tests were not difficult to make, but considerable time and patience were required to obtain a fair degree of accuracy. Such factors as the length of the pendulum arms, the attitude of the bodies, and timing technique were critical, especially for the small bodies. The calculated results depended largely on small differences between relatively large squared values. This being the case, it was necessary to take the average of many readings in order to obtain reliable data.

PERTINENT FORMULAS

The following symbols apply to the formulas used for calculating moments of inertia, apparent additional mass, and apparent additional mass coefficients:

a = length of cylinder (height)

B = object at end of pendulum rod or arm

I = increase in mass moment of inertia due to a single object $\underline{B}$ moving in water

I_a = mass moment of inertia in air of the cross-arm with both objects attached

I_o = true mass moment of inertia as measured in a vacuum

I_k = mass moment of inertia of pendulum arm and attached body used to determine the torsional constant of the wire pendulum

I_w = mass moment of inertia in water of the cross-arm with both models attached

I_{x_o} = mass moment of inertia of a cylinder about its x-axis through its center of gravity

I_{y_o} = mass moment of inertia of a cylinder about its y-axis through its center of gravity

I_a' = mass moment of inertia in air of cross-arms without bodies

I_w' = mass moment of inertia in water of cross-arms without bodies

K_T = torsional constant of the wire

L = distance from the center of the wire to the center of gravity of an object $\underline{B}$

M = mass of a body

m_A = apparent additional mass of one object $\underline{B}$

$m_{D.F.}$ = mass of displaced fluid

r = radius of cylinder whose length is $\underline{a}$

T_a = period of pendulum in air with bodies attached

T_w = period of pendulum in water with bodies attached

T_k = period of oscillation in air of body whose mass moment of inertia in I_k

T_a' = period of pendulum in air without bodies

T_w' = period of pendulum in water without bodies

The period of a simple torsion pendulum in air is:[22]

$$T_a = 2\pi(I_a/K_T)^{\frac{1}{2}} . \tag{1}$$

Neglecting damping due to viscosity, the period in water is:[23]

$$T_w = 2\pi(I_w/K_T)^{\frac{1}{2}} . \tag{2}$$

The increase in moment of inertia due to a single object B moving in water is:

$$I = \tfrac{1}{2}\left[I_w - I_a - (I_w' - I_a')\right] . \tag{3}$$

When L is relatively long:

$$I = m_A L^2 . \tag{4}$$

The above expressions may be used to calculate both the apparent increase in moment of inertia and the apparent additional mass. However, it is more convenient to calculate the apparent additional mass in terms of the known periods and constants as follows:

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[22] Yee-Tak Yu, loc. cit.

[23] Loc. cit.

from (4),

$$m_A = I/L^2 \ ;$$

from (3),

$$I = \tfrac{1}{2}\left[I_w - I_a - (I_w' - I_a')\right]$$

and

$$m_A = \frac{\tfrac{1}{2}\left[I_w - I_a - (I_w' - I_a')\right]}{L^2} \ ;$$

from (1) and (2),

$$T_a = 2\pi(I_a/K_T)^{\frac{1}{2}} \ ,$$

$$T_w = 2\pi(I_w/K_T)^{\frac{1}{2}} \ ,$$

$$K_T = 4\pi^2(I_k/T_k^2).$$

Making substitutions and simplifying:

$$I_a = \frac{T_a^2 K_T}{4\pi^2} \ ;$$

$$I_w = \frac{T_w^2 K_T}{4\pi^2} \ ;$$

$$I_a' = \frac{T_a'^2 K_T}{4\pi^2} \ ;$$

$$I_w' = \frac{T_w'^2 K_T}{4\pi^2} \ ;$$

$$m_A = \frac{I}{L^2} = \frac{\frac{1}{2}\left[I_w - I_a - (I_w' - I_a')\right]}{L^2},$$

$$= \frac{\frac{1}{2}\left[\frac{T_w^2 K_T}{4\pi^2} - \frac{T_a^2 K_T}{4\pi^2} - \left(\frac{T_w'^2 K_T}{4\pi^2} - \frac{T_a'^2 K_T}{4\pi^2}\right)\right]}{L^2},$$

$$= \frac{\frac{\frac{1}{2} K_T}{4\pi^2}\left[T_w^2 - T_a^2 - (T_w'^2 - T_a'^2)\right]}{L^2},$$

$$= \frac{\frac{1}{2}\left(\frac{1}{4\pi^2}\right)\left[4\pi^2 \frac{I_k}{T_k}\right]\left[T_w^2 - T_a^2 - (T_w'^2 - T_a'^2)\right]}{L^2},$$

$$= \frac{I_k}{2T_k^2 L^2}\left[T_w^2 - T_a^2 - (T_w'^2 - T_a'^2)\right]. \tag{5}$$

Equation (5) eliminates the necessity for calculating K_T, and the expression $\frac{I_k}{2T_k^2L^2}$ is usually constant for a complete series of tests. The usual formulas for mass moments of inertia may be used to calculate I_k.

For cylinders:

$$I_{x_o} = \frac{M\left[r^2 + \frac{a^2}{3}\right]}{4}; \tag{6}$$

$$I_{y_o} = \frac{M r^2}{2}; \tag{7}$$

$$I_k = I_{x_o} + M x^2$$

or

$$I_k = I_{y_o} + M y^2. \tag{8}$$

For relatively large masses at a distance from the wire the terms I_{x_o} and I_{y_o} become very small compared with the terms Mx^2 and My^2. Also the moments of inertia of the pendulum arms and clamps are very small compared with that of heavier bodies at a relatively great distance from the wire.

The formula for apparent additional mass coefficients is

$$k = \frac{m_A}{\text{mass of displaced fluid}}$$

$$= \frac{m_A}{m_{D.F.}} . \qquad (9)$$

Using the English system of units:

$$\text{Mass moment of inertia} = \text{slugs feet}^2$$

$$= \frac{\text{pounds seconds}^2 \text{ feet}^2}{\text{feet}}$$

$$= \text{pounds seconds}^2 \text{ feet.}$$

To convert slugs feet² to grams centimeters²:

$$\text{Slugs feet}^2 = 32.2 \ (454) \ (144) \ (2.54)^2 \text{ grams centimeters}^2$$

$$1 \text{ slug foot}^2 = 13,580,000 \text{ grams centimeters}^2. \qquad (10)$$

To convert slugs to grams:

$$1 \text{ slug} = 32.2 \ (454) \text{ grams} = 14,630 \text{ grams.} \qquad (11)$$

The uses of the formulas presented in this section are illustrated in Appendix I under the heading "Sample Calculations."

CONCLUSIONS

A study of the test results presented in this paper leads to the following conclusions.

1. As might be expected, in general, the apparent additional mass of a cylinder with blunt ends is greater than that of a similar cylinder with either conical or hemispherical noses. Likewise, the apparent additional mass of a cylinder with hemispherical ends is greater than that of a similar cylinder with conical ends.

2. Curves showing the variation of apparent additional mass with length for all the models tested are essentially straight lines for the range of lengths tested.

3. It is unlikely in some cases and impossible in others for the curves of apparent additional mass plotted against length to remain straight lines as the lengths approach zero. Evidently, there is either a sudden break or more gradual departure from a straight line as the fineness ratios become small. These breaks or gradual departures from straight lines appear to take place at fineness ratios less than two.

4. The attitude of the cylinder with respect to its direction of motion has a great effect on its apparent additional mass.

5. The shapes of the apparent additional mass coefficient curves at low fineness ratios are not very reliable. Due to the small magnitudes involved, very small changes in apparent additional mass values produce great changes in the values of the corresponding coefficients.

6. Apparent additional mass coefficients vary appreciably and are of practical importance for all the bodies tested as the fineness

ratios approach ten, especially for vertical attitudes.

7. As the fineness ratios increase the apparent additional mass coefficients tend to approach constant values. This fact emphasizes that only for large fineness ratios can the apparent additional mass coefficients be assumed constant.

8. Values of apparent additional mass coefficients for all the cylinders tested, for motion perpendicular to longitudinal axes, are greater than 1, and this is in line with the theoretical value of 1. Since the theoretical value is based on perfect fluid, test values should be greater than 1 due to viscosity.

9. Due to the absence of both theoretical and empirical values of apparent mass coefficients for cylinders moving parallel to longitudinal axes no comparisons with the values presented herein are possible. However, the author believes that the data presented are at least qualitatively indicative of the apparent additional mass effects for the bodies tested.

10. The torsion pendulum method appears to be a reliable and convenient test apparatus for determining the apparent additional mass characteristics of the bodies tested.

RECOMMENDATIONS

Among the important apparent additional mass studies needed to furnish empirical data on the subject are the following.

1. Scale tests are particularly timely. There is very little literature on this phase of the subject.

2. Investigations to determine the exact shape of the curves for apparent additional mass or apparent additional mass coefficients against fineness ratios in the range of fineness ratios from zero to 3 or 4 would be valuable. The tests presented herein do not adequately cover this range.

3. Velocity effects are not covered at all by the investigations presented by the author. However, the torsion pendulum may not lend itself to this type of experiment.

4. Due to the scarcity of all types of data in the field of apparent additional mass tests similar to those made by the author would provide a check on the results presented.

5. The effect of polished surfaces has been only slightly investigated.

6. An improved method of adjusting the lengths of the pendulum arms and an electrical timing system would probably insure more accurate results were the tests presented herein repeated. This is especially true for low fineness ratios. There is the possibility that unstable flow conditions prevent accurate tests at these low fineness ratios.

7. The effects of wall boundaries would be a valuable field of investigation. Exact measurements would be necessary.

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APPENDIX I

SAMPLE CALCULATIONS

APPENDIX I

SAMPLE CALCULATIONS

In order to determine the effects the arms and clamp of the torsion pendulum apparatus on the mass moment of inertia of the complete pendulum with bodies attached, it was necessary to make separate calculations for each part of the system.

For the two steel cylinders used to calibrate the pendulum:

Mass = M

= 583.6 grams = 1.284 pounds = 0.0399 slugs for 2 cylinders,

a = 3 inches = 0.25 feet,

r = 0.5 inches = 0.0416 feet.

$$I_k = I_{x_o} + ML^2 = \frac{M\left[r^2 + \frac{a^2}{3}\right]}{4} + ML^2 ,$$

$$= \frac{M}{4}\left[(0.0416)^2 + \frac{(0.25)^2}{3}\right] + ML^2 ,$$

$$= 0.0002325 + 0.0399\ L^2 = 0.0002325 + 0.0278325$$

$$= 0.027835 \text{ pounds seconds}^2 \text{ feet (for } L = 10 \text{ inches} = 0.8325 \text{ feet).}$$

For 1 brass cylindrical rod (8 inches long and 3/16 inch in diameter):

M = 28.5 grams = 0.00196 slugs,

x = 4.25 inches = 0.354 feet,

a = 8.00 inches = 0.666 feet.

$$I_{x_o}\ (\text{rod}) = \frac{M}{4}\left[r^2 + \frac{a^2}{3}\right] = \frac{M}{4}\left[0.000244 + 0.1484\right]$$

$$= 0.00196\ (0.037161)$$

$$= 0.0000728 \text{ pounds seconds}^2 \text{ feet.}$$

$$I_k\ (\text{rod}) = 0.0000728 + Mx^2 = 0.0000728 + 0.00196\ (0.354)^2$$

$$= 0.0000728 + 0.000151$$

$$= 0.0002238 \text{ slugs feet}^2.$$

For clamp (considered a cylinder):

$$M = 54.0 \text{ grams} = 0.00372 \text{ slugs},$$

$$a = 1.00 \text{ inch} = 0.0834 \text{ feet},$$

$$r = 0.50 \text{ inch} = 0.0417 \text{ feet}.$$

$$I_{x_o}\ (\text{clamp}) = \frac{M}{4}\left[r^2 + \frac{a^2}{3}\right] = \frac{M}{4}\left[(0.0417)^2 + \frac{(0.0834)^2}{3}\right]$$

$$= \frac{M}{4}\ (0.00174 + 0.00232) = 0.00093\ (0.00406)$$

$$= 0.00000377 \text{ slugs feet}^2.$$

$$I_k\ (\text{clamp}) = I_{x_o}\ (\text{clamp}) = 0.00000377 \text{ slgus feet}^2.$$

For complete pendulum:

$$I_k\ (2 \text{ cylinders}) = 0.02783500 \text{ slugs feet}^2$$

$$I_k\ (2 \text{ rods}) = 0.00044760 \text{ slugs feet}^2$$

$$I_k\ (\text{clamp}) = 0.00000377 \text{ slugs feet}^2$$

$$I_k\ (\text{pendulum}) = 0.02828487 \text{ slugs feet}^2$$

Error due to neglecting clamp $= \dfrac{0.00000377}{0.02828487} = 0.01332\%.$

$$\text{Error due to neglecting rods} = \frac{0.00044760}{0.02828487} = 1.582\%.$$

The above calculations indicate that any approximations made in determining the mass moment of inertia of the complete pendulum could only slightly affect the accuracy of the results of the tests. Of course, the rods and clamp were not neglected, but it was necessary to assign an approximate shape to the pendulum clamp and account for adjustment screws.

The following example illustrates the method used to calculate the apparent additional masses of the models tested. In all cases the periods were obtained by taking averages of many readings, ten or more in most cases.

For a solid aluminum cylinder with blunt ends, longitudinal axis parallel to motion, the apparent additional mass can be calculated from the following observed data:

Length = 4.50 inches	$T_{w'} = 1.160$ seconds
Diameter = 0.75 inches	$T_{a'} = 1.094$ seconds
$T_w = 3.81$ seconds	$T_k = 7.22$ seconds
$T_a = 3.60$ seconds	$L = 7.80$ inches

Two steel cylinders were used to calibrate the pendulum, the data recorded being:

$L = 10.00$ inches $= 0.8325$ feet,

$M = 0.0399$ slugs (complete pendulum),

$r = 0.50$ inches $= 0.0416$ feet,

$a = 3.00$ inches $= 0.25$ feet.

$$I_{x_o} = \frac{M}{4}\left[r^2 + \frac{a^2}{3}\right] = \frac{0.0399}{4}\left[(0.0416)^2 + \frac{(0.25)^2}{3}\right]$$

$$= 0.0002325 \text{ slugs feet}^2.$$

$$I_k = I_{x_o} + ML^2 = 0.0002325 + 0.0399\ (0.8325)^2$$

$$= 0.02783 \text{ slugs feet}^2.$$

$$m_A = \frac{I_k}{2T_k^2\ L^2}\left[T_w^{\ 2} - T_a^{\ 2} - (T_{w'}^{\ 2} - T_{a'}^{\ 2})\right]$$

$$= \frac{0.02783}{2(7.22)^2\ (7.8/12)^2}\left[14.53 - 13.00 - (1.35 - 1.20)\right]$$

$$= 0.00063\ (1.53 - 0.15) = 0.00063\ (1.38)$$

$$= 0.000870 \text{ slugs} = 12.73 \text{ grams.}$$

Since I_k, T_k, and $\underline{L}$ are usually constant for a series of tests, the calculations are essentially the process of finding the differences of the squared terms in the brackets of the formula for m_A and multiplying by a constant.

In order to calculate the apparent additional mass coefficient it is necessary to calculate the mass of the displaced fluid. For the 24 ST aluminum cylinders the specific gravity is 2.79.

$$\text{Displaced volume} = \text{mass}/2.79 = 91.20/2.79 = 32.70 \text{ cm.}^3$$

Therefore the displaced mass of fluid ($m_{D.F.}$) = 32.70 grams. Thus

$$k = m_a/m_{D.F.} = 12.73/32.70 = 0.389.$$

The data presented in Tables I through X of Appendix II were obtained from similar calculations.

APPENDIX II

TABLES

TABLE I

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING PARALLEL TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. x LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	1.950	1.74	0.65	7.22	0.896	0.01242
0.75 x 1.50	2.440	2.24	0.65	7.22	0.896	0.01513
0.75 x 2.00	2.730	2.54	0.65	7.22	0.896	0.01741
0.75 x 2.25	2.835	2.64	0.65	7.22	0.896	0.01835
0.75 x 3.00	3.218	3.02	0.65	7.22	0.896	0.02159
0.75 x 4.50	3.810	3.60	0.65	7.22	0.896	0.02800
0.75 x 6.00	4.360	4.14	0.65	7.22	0.896	0.03420
0.75 x 7.50	4.680	4.68	0.65	7.22	0.896	0.04090

TABLE II

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING PERPENDICULAR TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	1.978	1.72	0.65	7.22	0.896	0.01610
0.75 x 1.50	2.590	2.22	0.65	7.22	0.896	0.03285
0.75 x 2.00	2.950	2.52	0.65	7.22	0.896	0.04465
0.75 x 2.25	3.030	2.64	0.65	7.22	0.896	0.04835
0.75 x 3.00	3.510	3.00	0.65	7.22	0.896	0.06500
0.75 x 4.50	4.210	3.57	0.65	7.22	0.896	0.09825
0.75 x 6.00	4.830	4.08	0.65	7.22	0.896	0.13200
0.75 x 7.50	5.350	4.50	0.65	7.22	0.896	0.16550

TABLE III

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	1.950	1.73	0.65	7.22	0.896	0.0132
0.75 x 1.50	2.510	2.23	0.65	7.22	0.896	0.0237
0.75 x 2.00	2.830	2.53	0.65	7.22	0.896	0.0294
0.75 x 2.25	2.941	2.62	0.65	7.22	0.896	0.0335
0.75 x 3.00	3.360	3.01	0.65	7.22	0.896	0.0425
0.75 x 4.50	4.010	3.59	0.65	7.22	0.896	0.0636
0.75 x 6.00	4.610	4.12	0.65	7.22	0.896	0.0845
0.75 x 7.50	5.110	4.58	0.65	7.22	0.896	0.1050

TABLE IV

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING PARALLEL TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	-----	----	0.65	7.22	0.896	-------
0.75 x 1.50	2.027	1.95	0.65	7.22	0.896	0.00611
0.75 x 2.00	-----	----	0.65	7.22	0.896	-------
0.75 x 2.25	2.520	2.40	0.65	7.22	0.896	0.00852
0.75 x 3.00	2.878	2.76	0.65	7.22	0.896	0.01073
0.75 x 4.50	3.560	3.43	0.65	7.22	0.896	0.01520
0.75 x 6.00	4.180	4.04	0.65	7.22	0.896	0.02028
0.75 x 7.50	4.680	4.54	0.65	7.22	0.896	0.02510

TABLE V

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING PERPENDICULAR TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	-----	----	0.65	7.22	0.896	-------
0.75 x 1.50	2.31	1.94	0.65	7.22	0.896	0.02900
0.75 x 2.00	-----	-----	0.65	7.22	0.896	-------
0.75 x 2.25	2.835	2.40	0.65	7.22	0.896	0.04300
0.75 x 3.00	3.26	2.78	0.65	7.22	0.896	0.05640
0.75 x 4.50	4.00	3.40	0.65	7.22	0.896	0.08650
0.75 x 6.00	4.62	3.94	0.65	7.22	0.896	0.11600
0.75 x 7.50	5.17	4.40	0.65	7.22	0.896	0.14500

TABLE VI

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

CYLINDER IN DIA. x LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	-----	----	0.65	7.22	0.896	-------
0.75 x 1.50	2.110	1.94	0.65	7.22	0.896	0.01020
0.75 x 2.00	-----	----	0.65	7.22	0.896	-------
0.75 x 2.25	2.640	2.40	0.65	7.22	0.896	0.02093
0.75 x 3.00	3.070	2.77	0.65	7.22	0.896	0.0320
0.75 x 4.50	3.792	3.42	0.65	7.22	0.896	0.05315
0.75 x 6.00	4.440	4.00	0.65	7.22	0.896	0.07200
0.75 x 7.50	4.96	4.48	0.65	7.22	0.896	0.0935

TABLE VII

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDER WITH HEMISPHERICAL ENDS MOVING PARALLEL TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	-----	----	0.65	7.22	0.896	-------
0.75 x 1.50	2.186	2.09	0.65	7.22	0.896	0.00486
0.75 x 2.00	-----	----	0.65	7.22	0.896	-------
0.75 x 2.25	2.635	2.53	0.65	7.22	0.896	0.00805
0.75 x 3.00	3.000	2.88	0.65	7.22	0.896	0.01113
0.75 x 4.50	3.660	3.52	0.65	7.22	0.896	0.01723
0.75 x 6.00	4.250	4.10	0.65	7.22	0.896	0.02320
0.75 x 7.50	4.800	4.62	0.65	7.22	0.896	0.02940

TABLE VIII

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDER WITH HEMISPHERICAL ENDS MOVING PERPENDICULAR TO LONGITUDINAL AXES IN WATER

CYLINDER IN.		T_w	T_a	L	T_k	I_k	m_A
DIA.	LGT.	SEC.	SEC.	FT.	SEC.	LBS. FT.^{2}	LBS.
0.75 x	0.75	----	----	0.65	7.22	0.896	-------
0.75 x	1.50	2.38	2.07	0.65	7.22	0.896	0.02475
0.75 x	2.00	----	----	0.65	7.22	0.896	-------
0.75 x	2.25	2.90	2.51	0.65	7.22	0.896	0.03960
0.75 x	3.00	3.34	2.86	0.65	7.22	0.896	0.05700
0.75 x	4.50	4.09	3.50	0.65	7.22	0.896	0.08860
0.75 x	6.00	4.70	4.00	0.65	7.22	0.896	0.12230
0.75 x	7.50	5.28	4.48	0.65	7.22	0.896	0.15520

TABLE IX

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS, m_A, OF SOLID ALUMINUM CYLINDERS WITH HEMISPHERICAL ENDS MOVING FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

CYLINDER IN. DIA. LGT.	T_w SEC.	T_a SEC.	L FT.	T_k SEC.	I_k LBS. FT.^{2}	m_A LBS.
0.75 x 0.75	-----	-----	0.65	7.22	0.896	-------
0.75 x 1.50	2.261	2.080	0.65	7.22	0.896	0.0129
0.75 x 2.00	-----	-----	0.65	7.22	0.896	-------
0.75 x 2.25	2.760	2.520	0.65	7.22	0.896	0.02285
0.75 x 3.00	3.170	2.870	0.65	7.22	0.896	0.03345
0.75 x 4.50	3.880	3.510	0.65	7.22	0.896	0.05210
0.75 x 6.00	4.480	4.040	0.65	7.22	0.896	0.07200
0.75 x 7.50	5.040	4.560	0.65	7.22	0.896	0.09175

TABLE X

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS COEFFICIENT, k, OF SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

L/D RATIO	MASS OF DISPLACED FLUID G.	PARALLEL m_A G.	PARALLEL k	PERPENDICULAR m_A G.	PERPENDICULAR k	FORTY-FIVE DEGREES m_A G.	FORTY-FIVE DEGREES k
1	5.38	5.64	1.048	7.00	1.310	6.00	1.115
2	10.82	6.87	0.635	14.50	1.340	10.78	0.994
3	16.35	8.34	0.510	22.40	1.370	15.20	0.924
4	21.90	9.81	0.448	29.95	1.370	19.30	0.882
6	32.70	12.73	0.389	44.95	1.375	28.80	0.880
8	43.50	15.61	0.359	60.00	1.380	38.20	0.879
10	54.40	18.58	0.342	75.10	1.380	47.60	0.875

TABLE XI

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS COEFFICIENT, k, OF SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

L/D RATIO	MASS OF DISPLACED FLUID G.	PARALLEL m_A G.	PARALLEL k	PERPENDICULAR m_A G.	PERPENDICULAR k	FORTY-FIVE DEGREES m_A G.	FORTY-FIVE DEGREES k
1	-----	-----	-----	-----	-----	-----	-----
2	7.50	2.780	0.371	13.18	1.756	4.64	0.618
3	12.88	3.865	0.300	19.52	1.520	9.50	0.738
4	18.30	4.835	0.264	25.60	1.400	14.53	0.795
6	29.30	6.900	0.236	39.35	1.342	23.70	0.808
8	40.30	9.210	0.229	52.60	1.307	32.70	0.812
10	51.30	11.40	0.222	65.80	1.284	42.40	0.827

TABLE XII

DATA FOR THE DETERMINATION OF THE APPARENT ADDITIONAL MASS COEFFICIENT, k, OF SOLID ALUMINUM CYLINDERS WITH HEMISPHERICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

L/D RATIO	MASS OF DISPLACED FLUID G.	PARALLEL m_A G.	PARALLEL k	PERPENDICULAR m_A G.	PERPENDICULAR k	FORTY-FIVE DEGREES m_A G.	FORTY-FIVE DEGREES k
1	-----	-----	-----	-----	-----	-----	-----
2	9.05	2.22	0.246	11.10	1.227	5.85	0.647
3	14.60	3.66	0.251	18.00	1.233	10.38	0.710
4	19.75	5.06	0.256	25.90	1.265	15.20	0.764
6	30.70	7.82	0.255	40.30	1.315	23.98	0.781
8	41.70	10.52	0.252	55.60	1.335	32.80	0.787
10	52.90	13.36	0.253	70.50	1.335	41.70	0.788

APPENDIX III

FIGURES

FIGURE 1

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

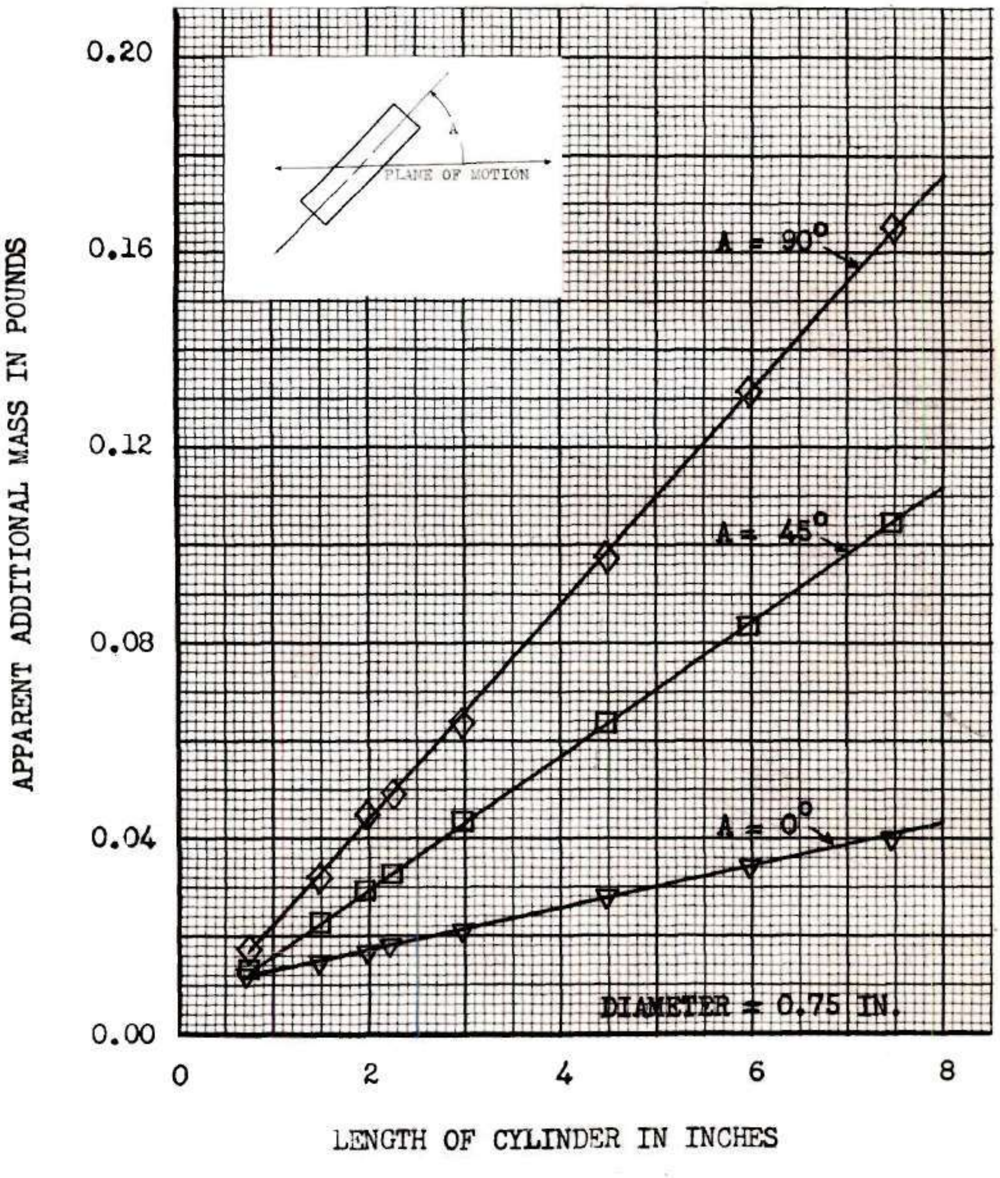


FIGURE 2

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

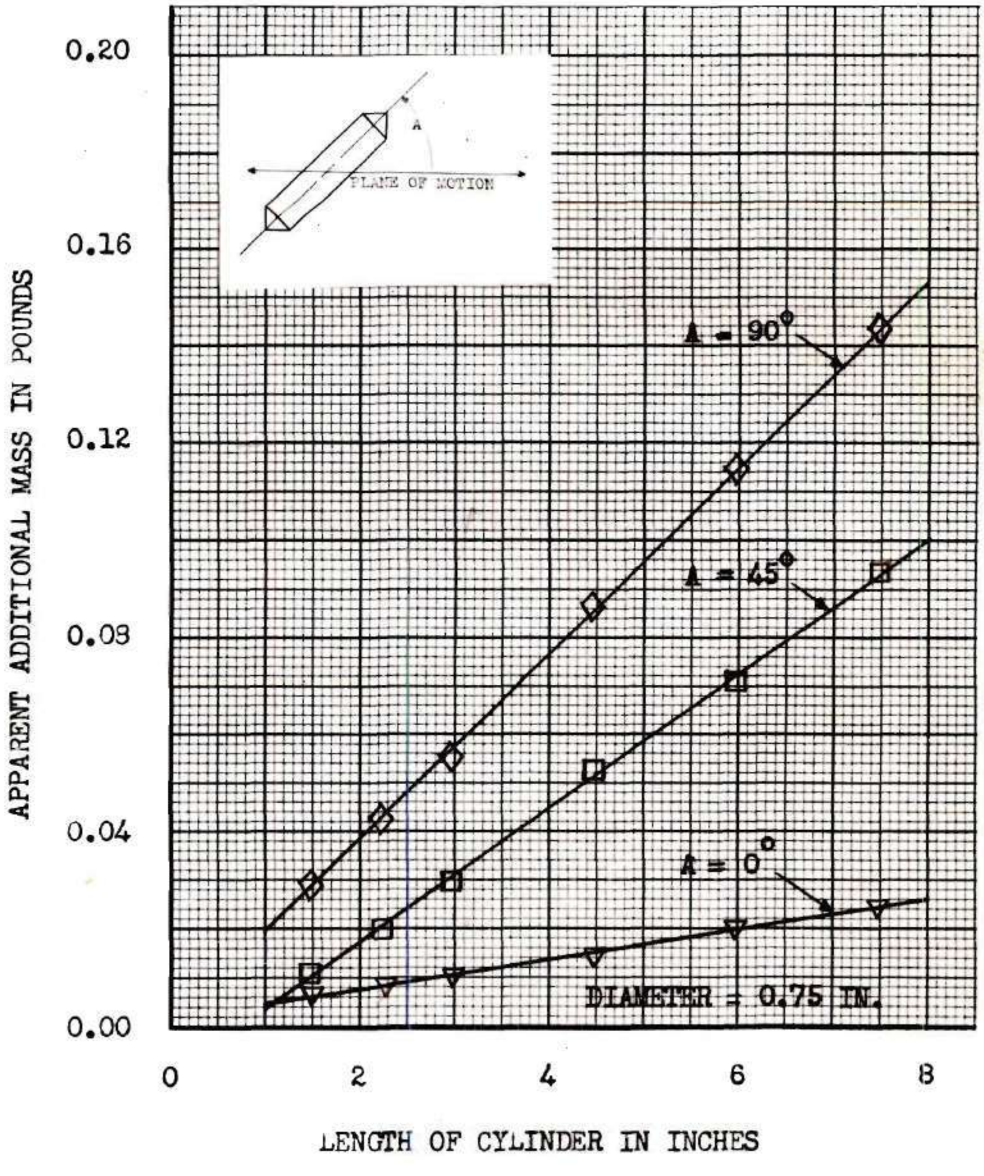


FIGURE 3

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH HEMISPHERICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

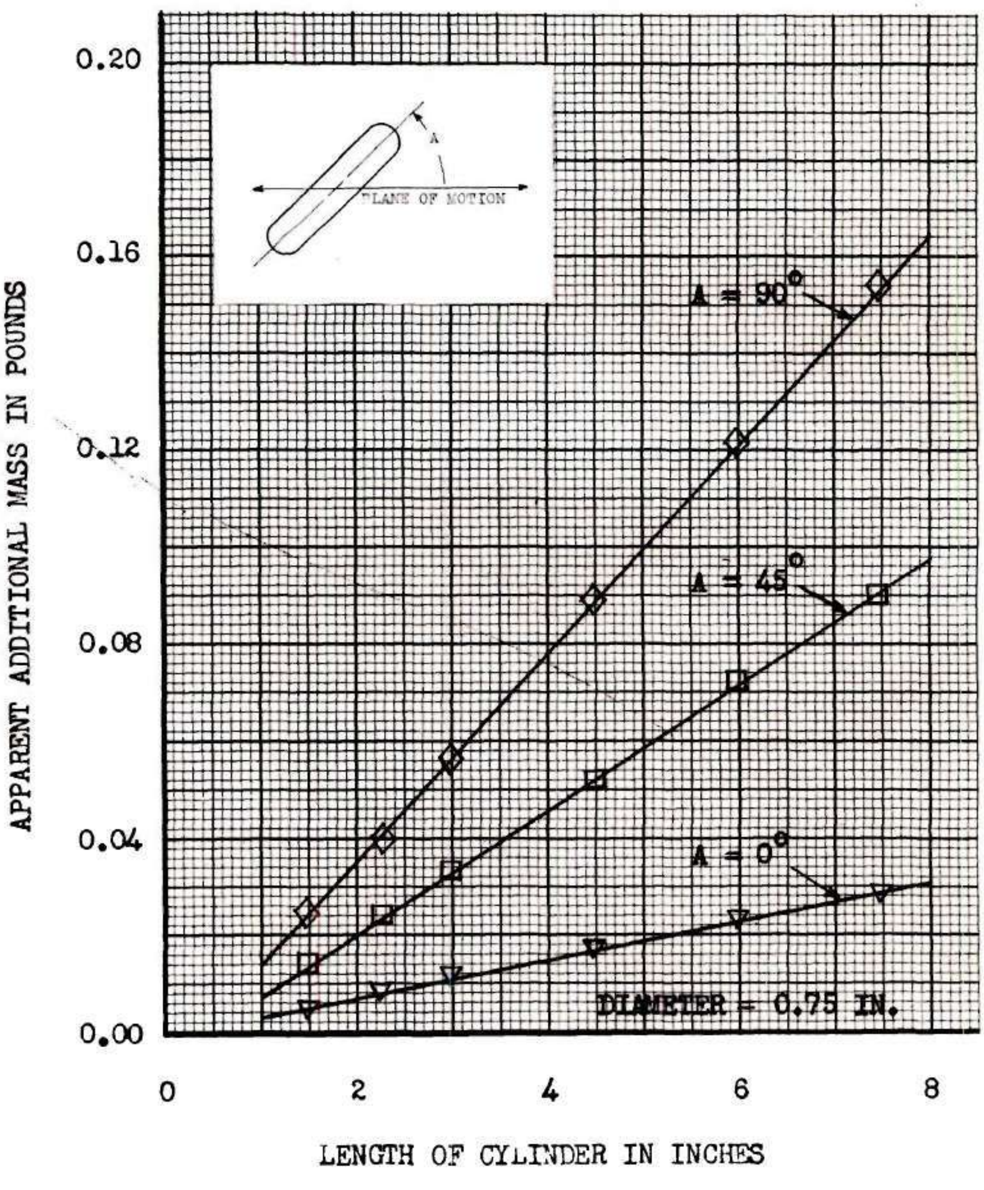


FIGURE 4

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH BLUNT, CONICAL, AND HEMISPHERICAL ENDS MOVING PERPENDICULAR TO LONGITUDINAL AXES IN WATER

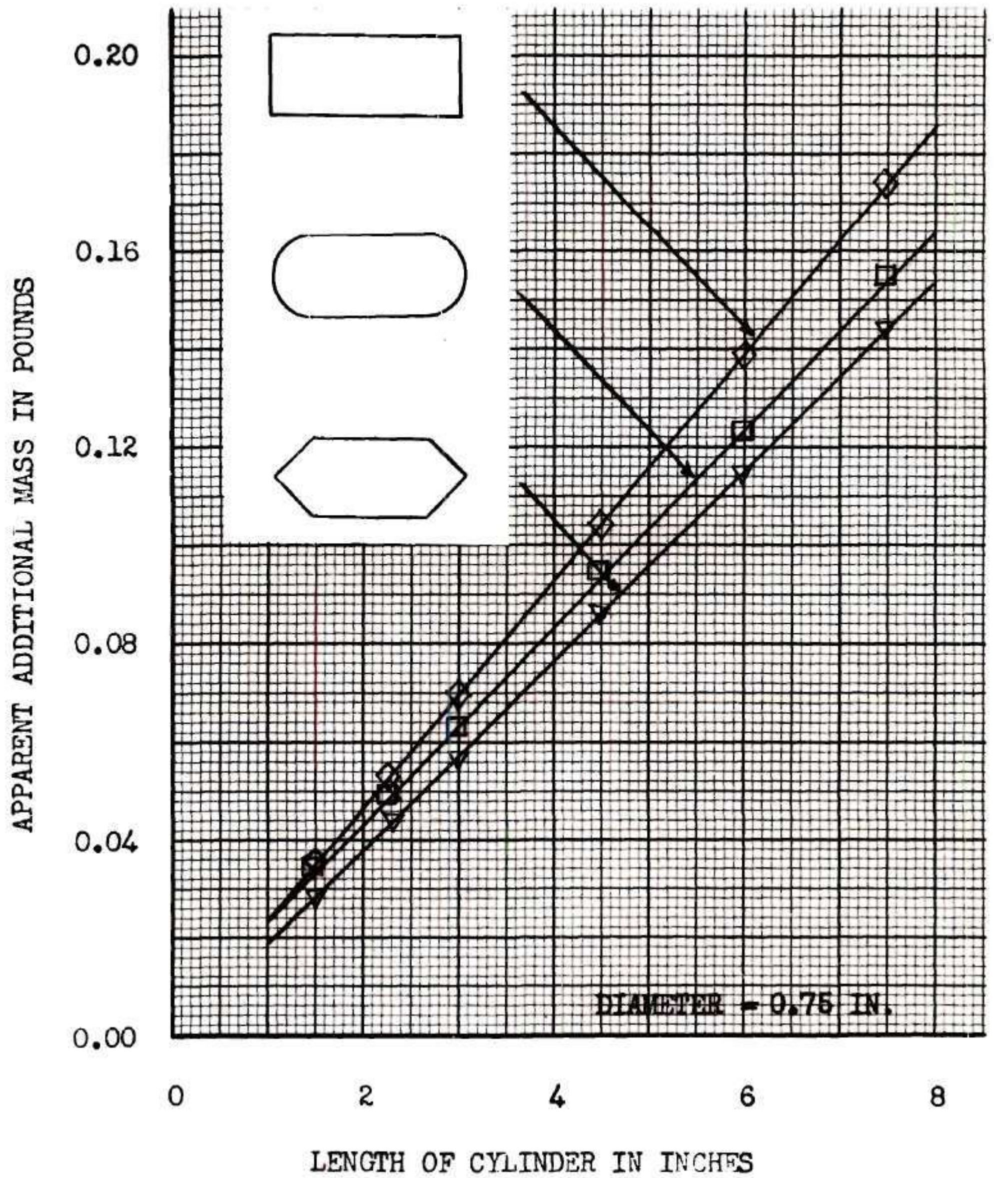


FIGURE 5

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH BLUNT, CONICAL, AND HEMISPHERICAL ENDS MOVING PARALLEL TO LONGITUDINAL AXES IN WATER

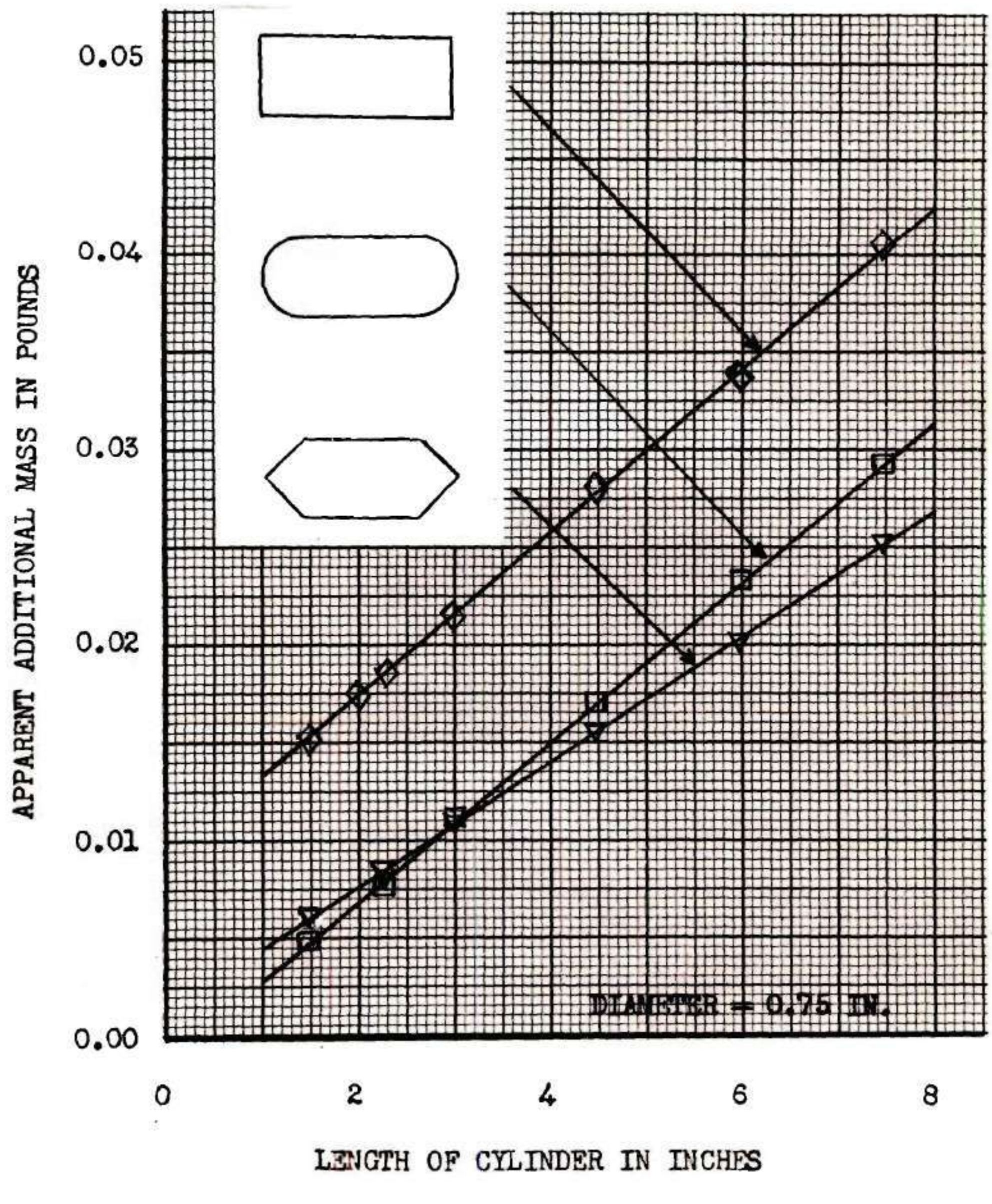


FIGURE 6

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR SOLID ALUMINUM CYLINDERS WITH BLUNT, CONICAL, AND HEMISPHERICAL ENDS MOVING FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

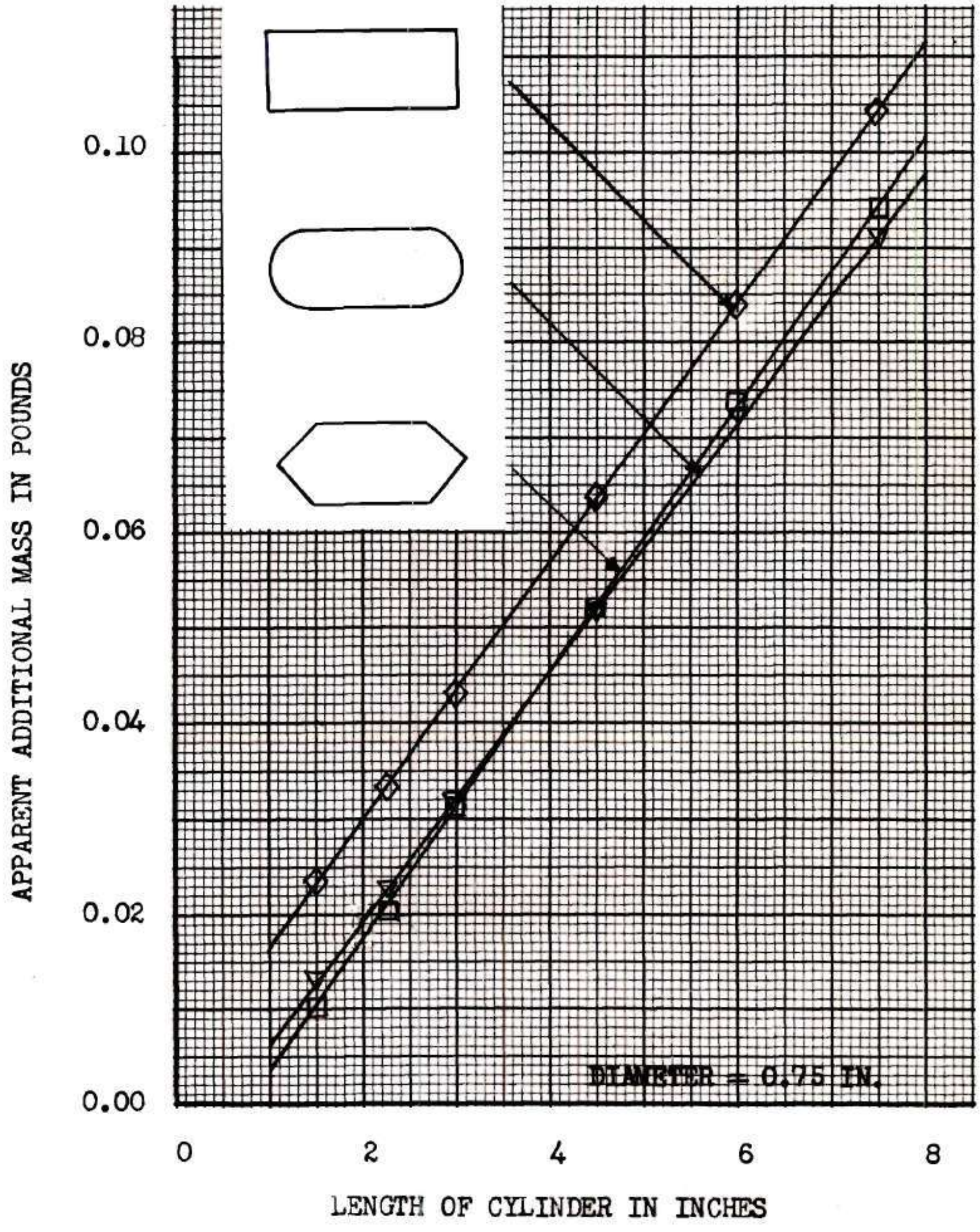


FIGURE 7

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS COEFFICIENT WITH FINENESS RATIO FOR SOLID ALUMINUM CYLINDERS WITH BLUNT ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

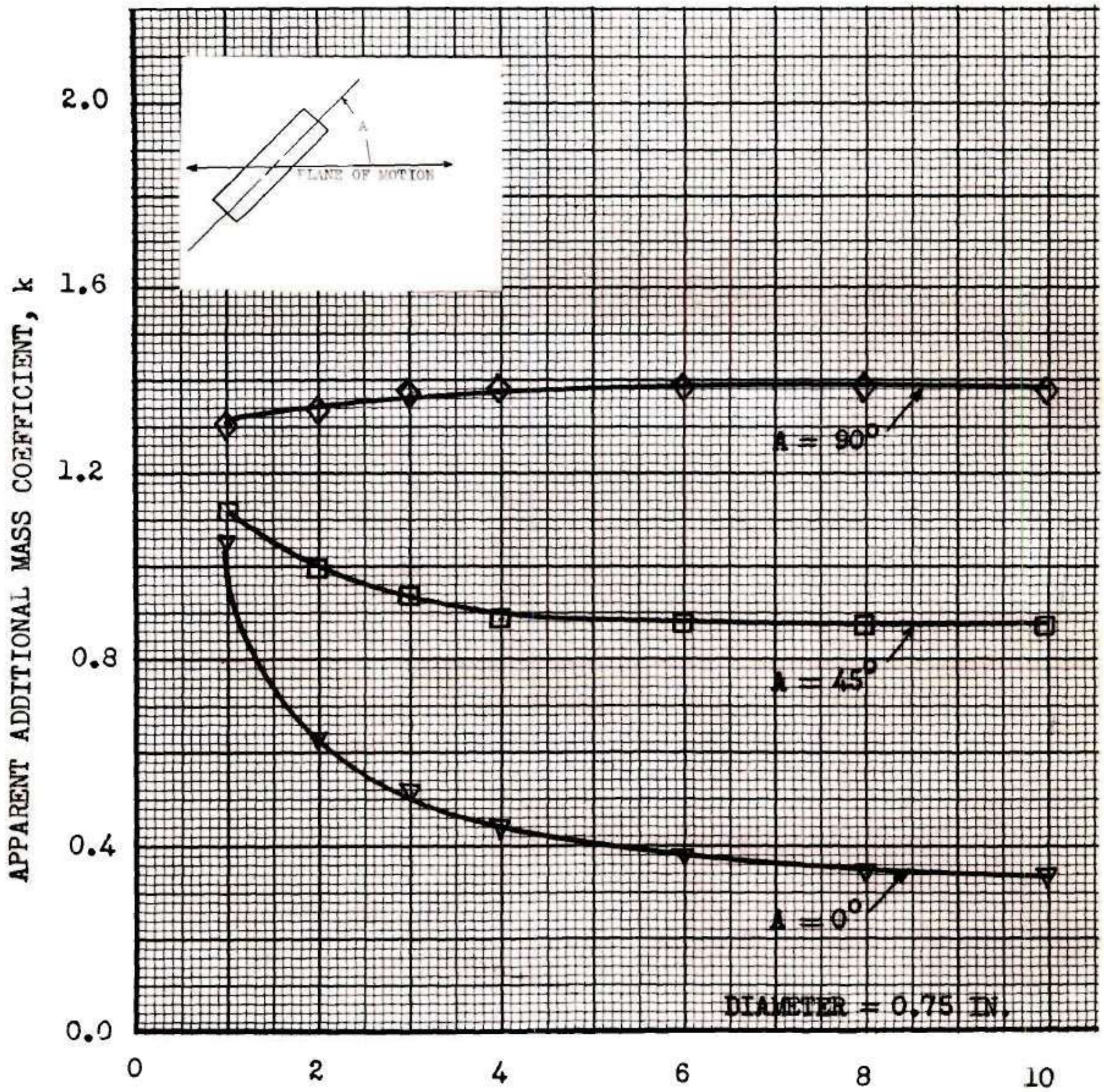


FIGURE 8

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS COEFFICIENT WITH FINENESS RATIO FOR SOLID ALUMINUM CYLINDERS WITH CONICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

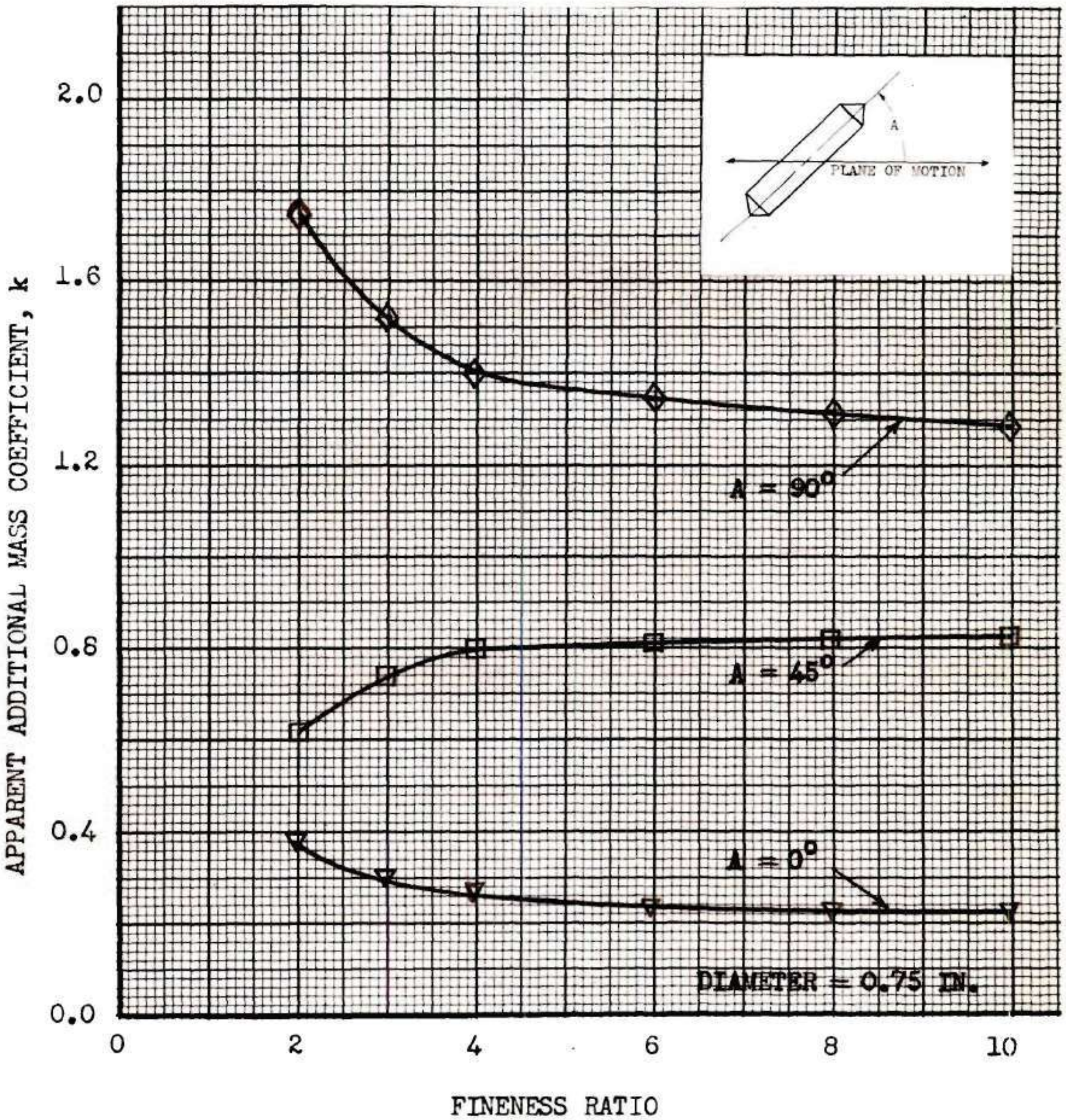


FIGURE 9

GRAPH SHOWING VARIATION OF APPARENT ADDITIONAL MASS COEFFICIENT WITH FINENESS RATIO FOR SOLID ALUMINUM CYLINDERS WITH HEMISPHERICAL ENDS MOVING PARALLEL, PERPENDICULAR, AND FORTY-FIVE DEGREES TO LONGITUDINAL AXES IN WATER

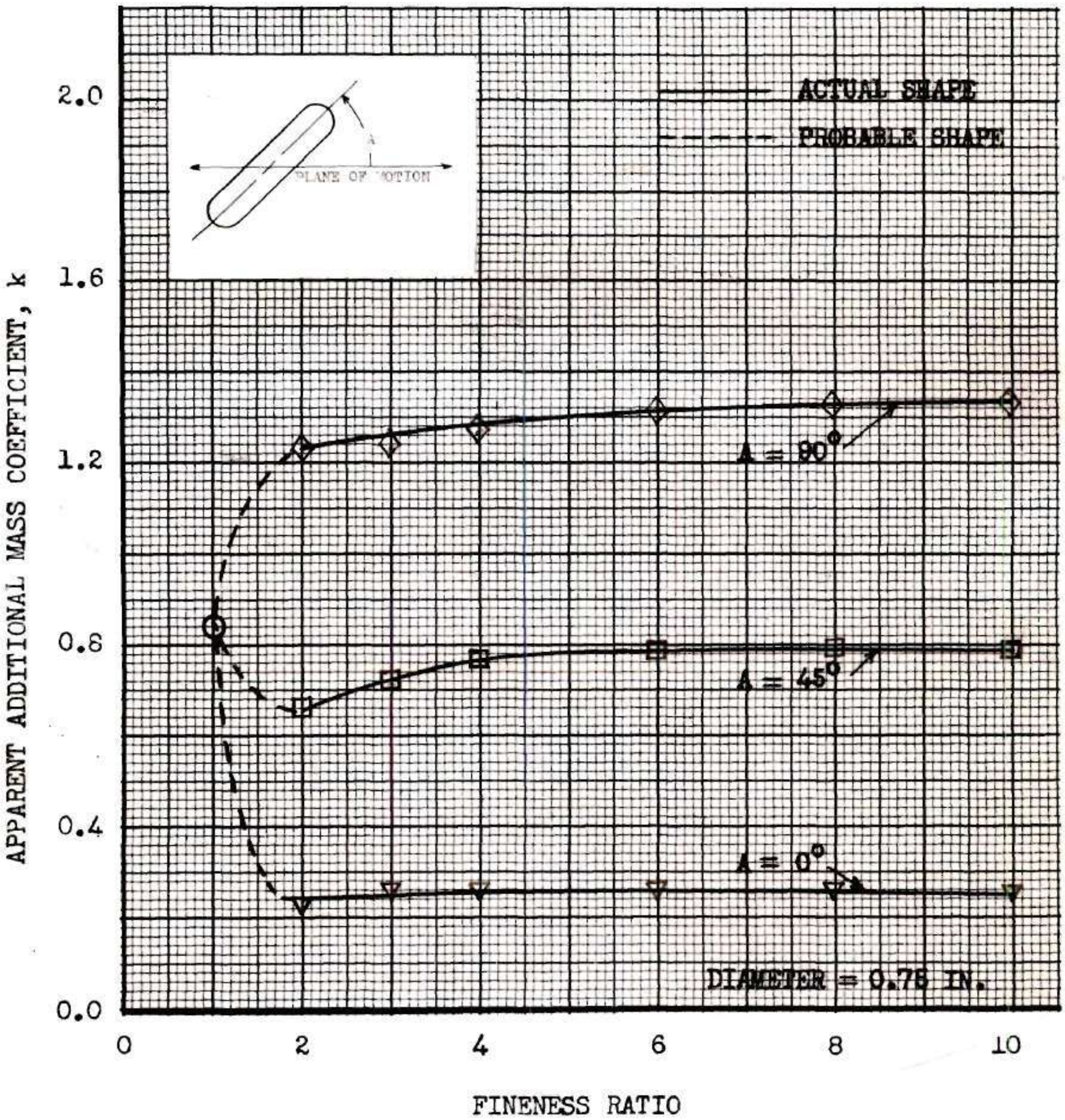
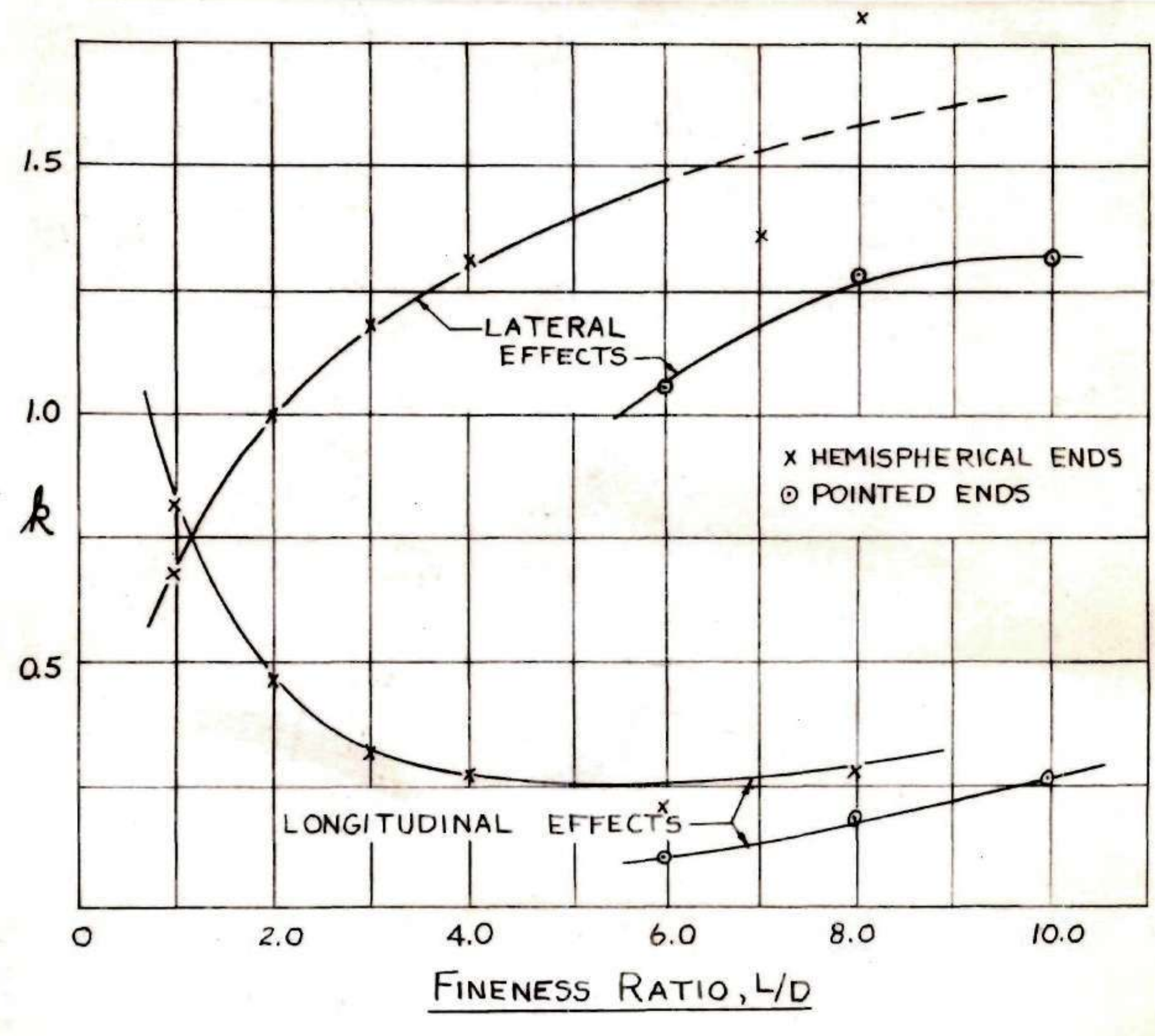
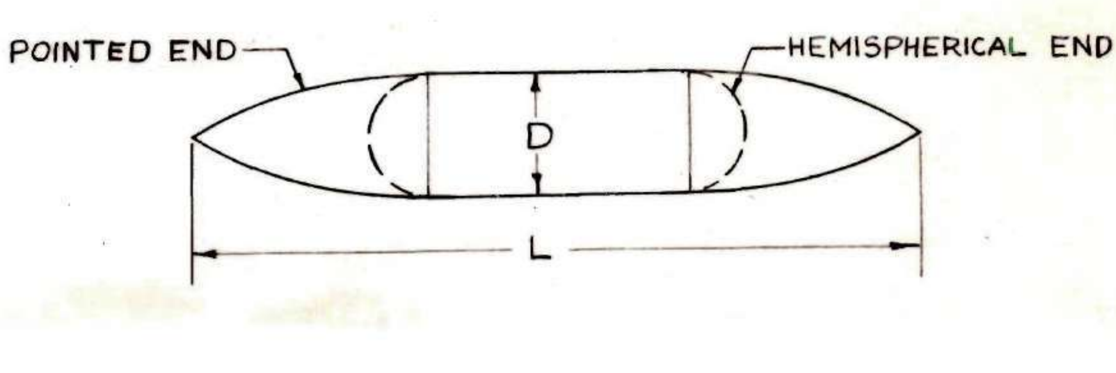


FIGURE 10

EFFECT OF FINENESS RATIO ON APPARENT ADDITIONAL MASS COEFFICIENTS

FROM DATA PRESENTED BY RELF AND JONES (REFERENCE 13)





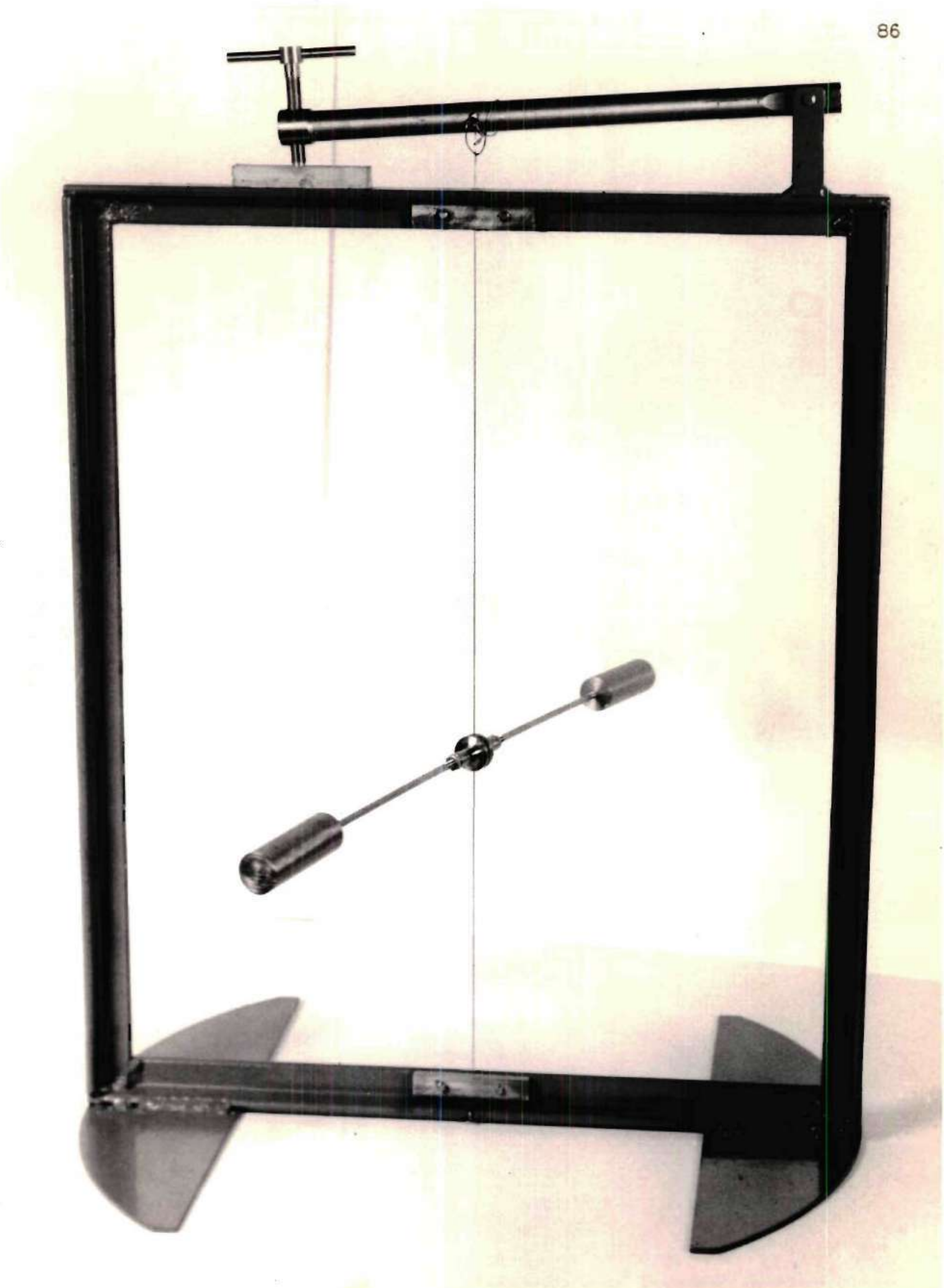


FIGURE 11

TORSION PENDULUM APPARATUS WITH STEEL CALIBRATION CYLINDERS ATTACHED

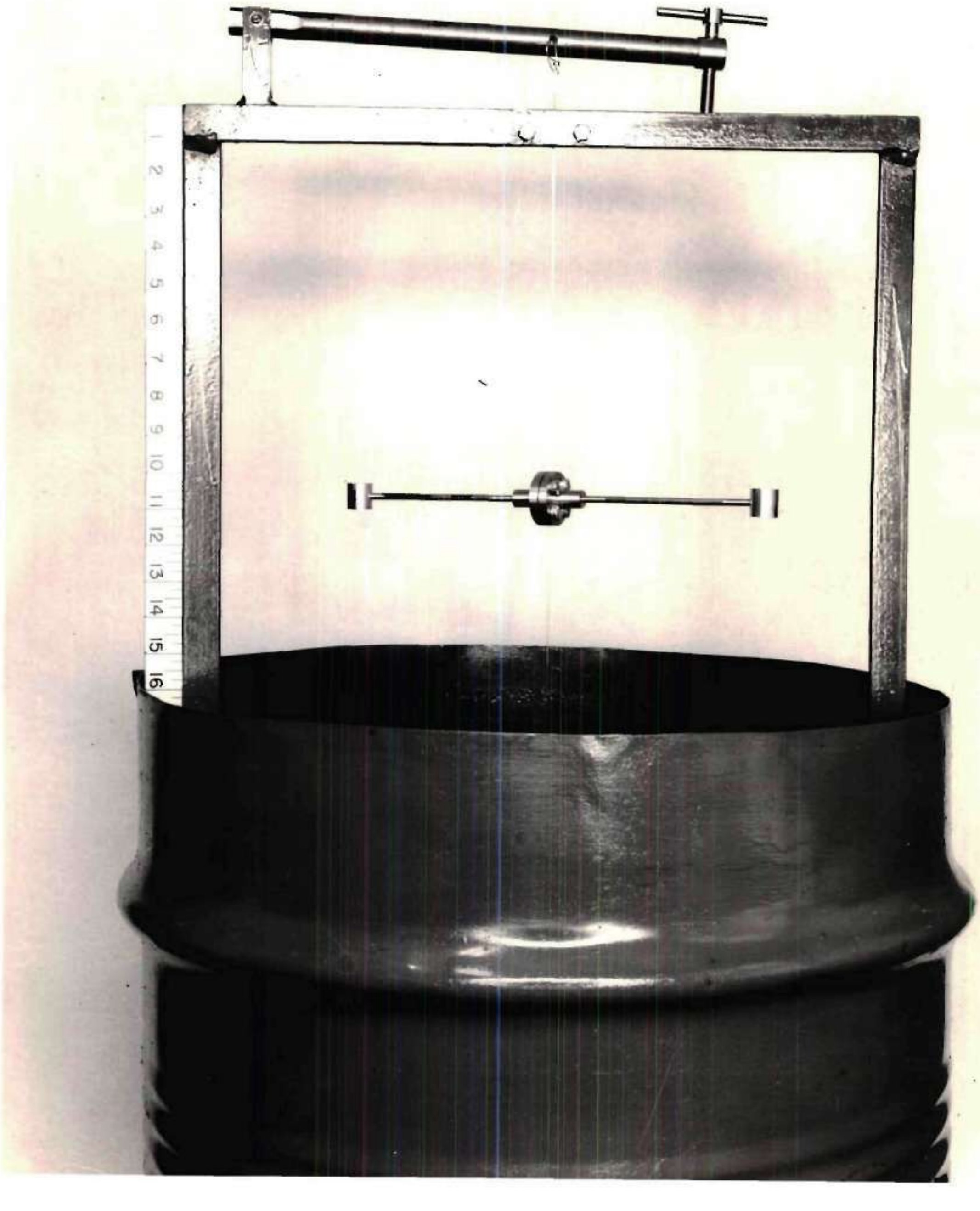


FIGURE 12

TORSION PENDULUM APPARATUS PARTIALLY SUBMERGED IN TANK

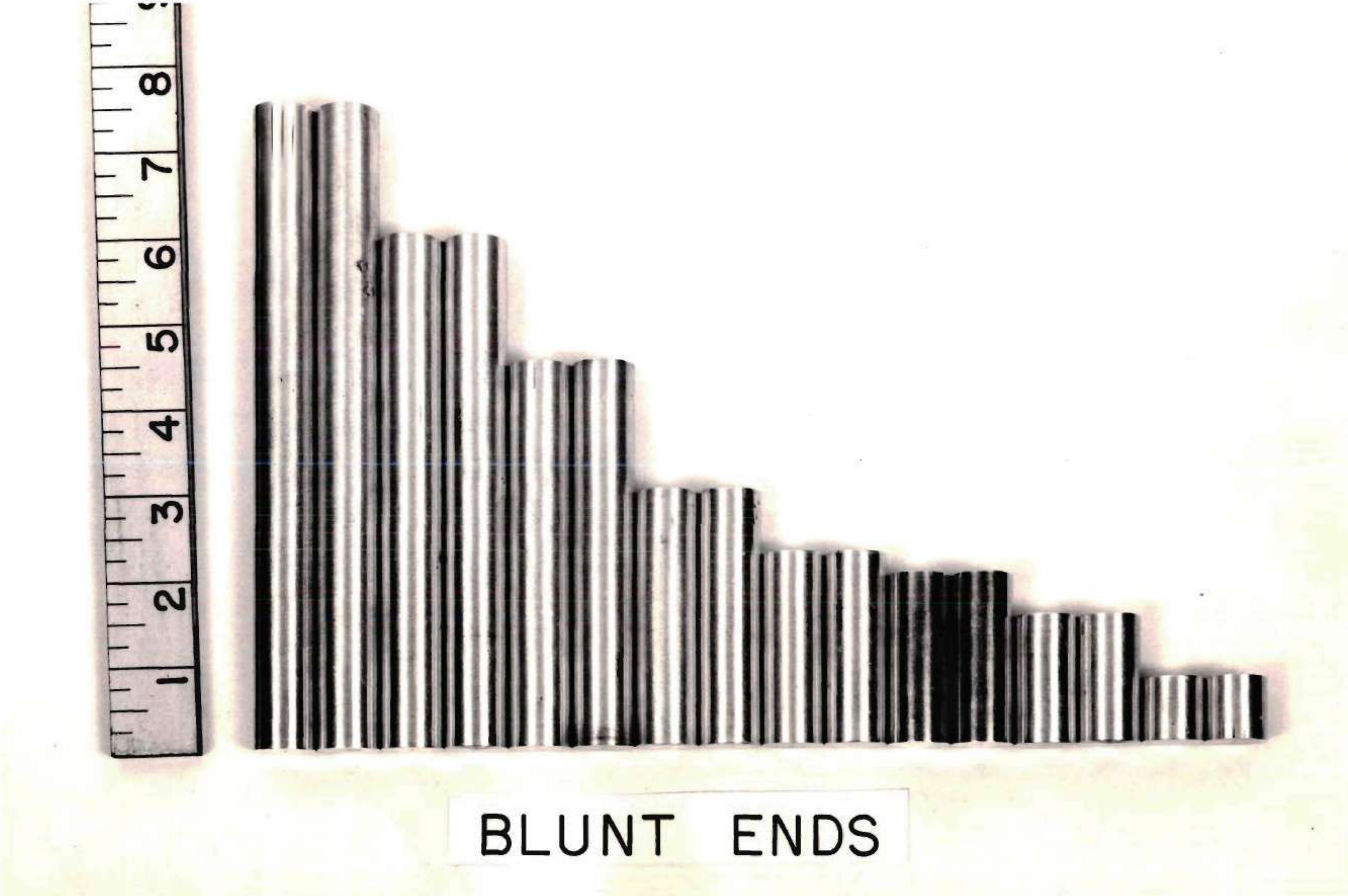


FIGURE 13

THE GROUP OF CYLINDRICAL MODELS WITH BLUNT ENDS USED IN THE EXPERIMENTAL INVESTIGATIONS

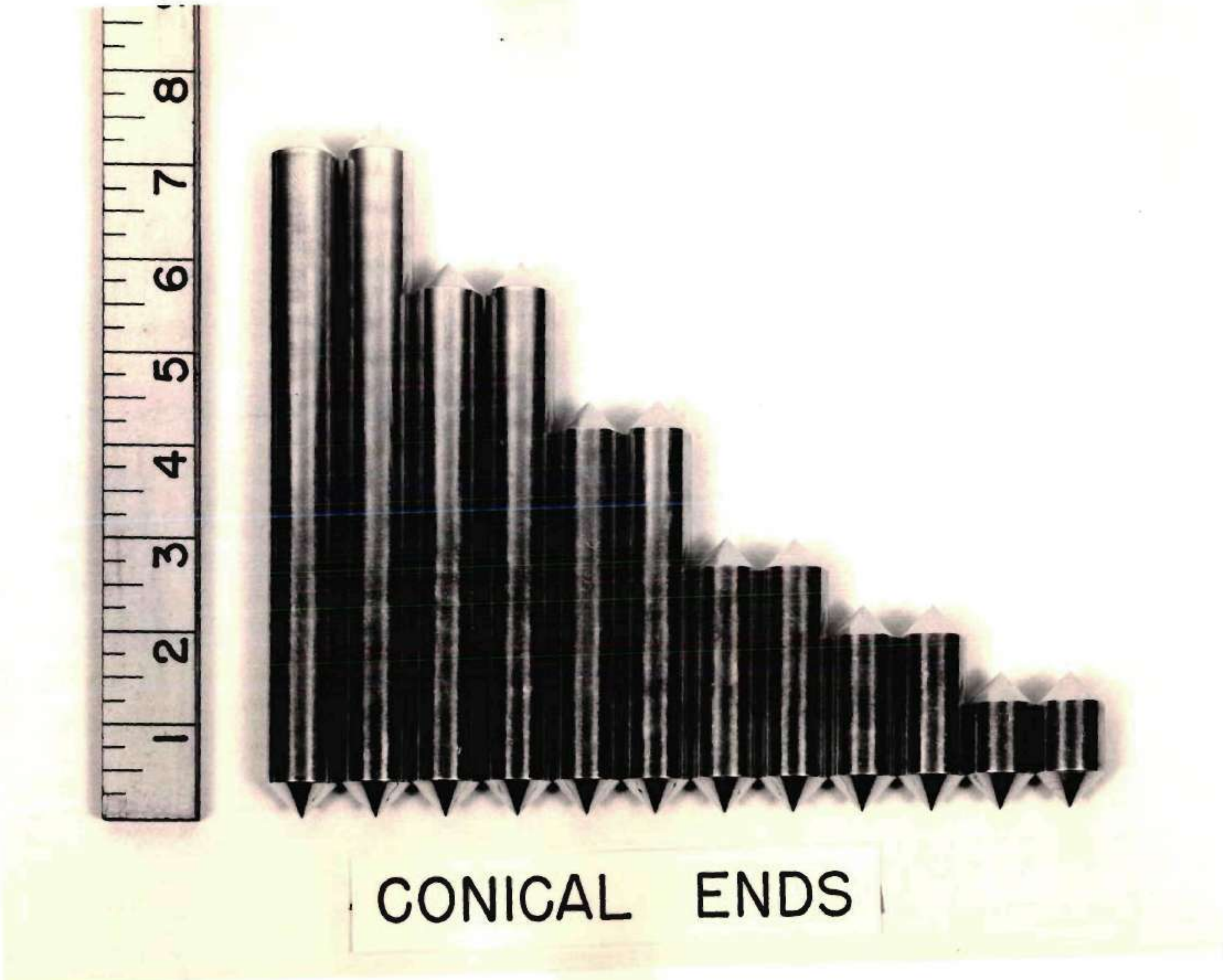


FIGURE 14

THE GROUP OF CYLINDRICAL MODELS WITH CONICAL ENDS USED IN THE EXPERIMENTAL INVESTIGATIONS

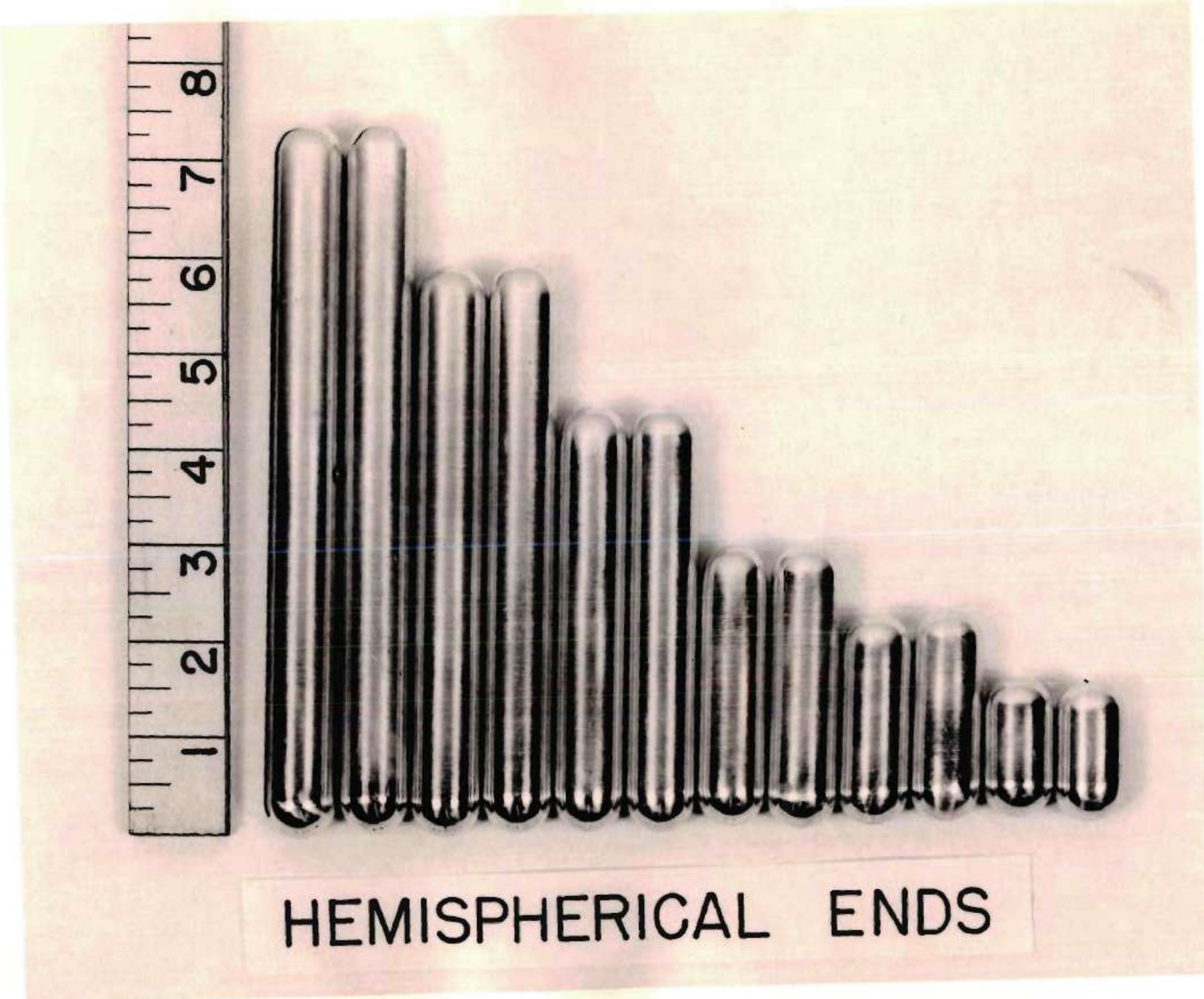


FIGURE 15

THE GROUP OF CYLINDRICAL MODELS WITH HEMISPHERICAL ENDS USED IN THE EXPERIMENTAL INVESTIGATIONS